QUANTUM TELEPORTATION

ON

GENERIC HILBERT SPACES

AND IN

OPTICS

MÁTYÁS KONIORCZYK

supervisor:

JÓZSEF JANSZKY

A dissertation
submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy
in physics

UNIVERSITY OF PÉCS

2002

Theoretical physicists live in a classical world
looking out into a quantum mechanical world.
The latter we describe only subjectively,
in terms of procedures and results
in our classical domain.

John Stuart Bell

Contents

Introduction

The end of the 20th century can be regarded as the second golden age of quantum mechanics, the core of all fields of modern physics. After its birth in the 1920's, it induced a revolution of physics, and became a frontier in the human thinking about the nature of reality. The theory was extremely successful: it yielded an appropriate model for the structure and interactions of matter, generation and absorption of light, and many other applications. There was a price to be paid for these achievements. Due to the more sophisticated mathematics employed, it has lost its understandability for everyday (non-physicist) people. On the other hand, even those who "speak" the language of mathematics still do not possess the opportunity of visualization or even imagination of phenomena in argument. And though quantum mechanics provides us with recipes to connect the abstract theory with the actual measurement results, it leaves a lot of counterintuitive ideas, interpretational or even more general philosophical problems, open questions behind. Most of these originate from two main concepts: quantum measurement and entanglement.

Were these problems, which are so much in the center of today's research interest, not strikingly visible from the very beginning? They were there definitely, but probably in spite of their extreme conceptual importance, they must have been regarded as marginal questions from a practical, experimental point of view. Some peculiar features of quantum entanglement for instance were very well understood in 1935 by Einstein, Podolsky and Rosen, but probably not many physicists thought of these as something one can see in the laboratory, or even apply for certain purposes. John Stuart Bell, who formulated the philosophical concept of nonlocality of quantum mechanics in mathematical terms, did this work as a hobby, and some of his earlier colleagues at CERN still got surprised hearing about his interest in this field. Even for John Clauser, the pioneer of Bell-experiments it was hard in the 1960's to find support for such investigations. These were considered as some interesting but unimportant problems of an otherwise successful theory that time.

From the works of D. Bohm and J. S. Bell, it is clear however, that "real quantum mechanics" begins with multipartite systems, as all the problems of single particle quantum mechanics can be treated in the framework of local hidden variable theories. Thus entanglement is indeed at the very heart of physics. It was in the 1980's, when experimental quantum optics reached a level, on which nonlocality and quantum measurement became an essential ingredient of understanding results.

The feasibility of entangled quantum states had another important consequence: the concept of quantum information was born. A new field of information and computation theory appeared. A theory of quantum communication and computation is being developed, which is a completely new perspective in cryptography, dense coding or solving problems that were unfeasible to treat with classical computers. The appearance of the concept of *information* in physics is also promising, especially in understanding fundamental problems of quantum mechanics. The non-existence of “quantum hardware”, a painful fact for quantum information scientists, provides experimental physics and technology with a great challenge.

This dissertation summarizes my results concerning quantum teleportation, a typical phenomenon based on entanglement, and the basic constituent of quantum communication. Since its first appearance in the paper of Ch. Bennett [7], it has attracted an extreme attention of both theoretical and experimental researchers. Though the idea itself is ingeniously simple, and may even have been suggested theoretically in the 1920’s, it has several features which are of definite current physical interest. It should become one of the fundamental building blocks of future quantum communication techniques. My work was consisting of understanding teleportation in general, finding alternative ways of its description, and searching for different physical systems for its realization.

The dissertation has the following structure. In order to make the discussions self-contained, Chapter 1 briefly summarizes those basic elements of quantum mechanics and quantum optics which will be applied in the subsequent chapters. Chapter 2 contains a short introduction to quantum teleportation: the description of two earliest protocols: the Bennett and the Braunstein-Kimble schemes.

Chapter 3 contains my results concerning the general description of teleportation on abstract Hilbert spaces, which may correspond to arbitrary physical systems. First the original Bennett scheme is shown to be a quantum generalization of the well-known “one-time-pad” cipher. A gedanken experiment is analyzed, revealing the classical-to-quantum transition. Then finite dimensional teleportation using an arbitrary bipartite pure state is investigated: a compact, basis independent expression is derived for the explicit form of quantum teleportation as a quantum operation. This is motivated by the relative-state representation of quantum channel theory. This topic is followed by a description of the Bennett scheme in terms of finite-phase-space Wigner functions, with an outline of the continuous dimensional limit. This description shows the connection between finite and continuous variable quantum teleportation in its clearest form.

In chapter 4 the results concerning continuous variable optical quantum teleportation are described. I describe the Braunstein-Kimble scheme in terms of low-dimensional coherent state representations. This is a novel approach providing an alternative, and rather simple understanding of the phenomenon.

In chapter 5, few-photon interference schemes are considered. A generalization of "quantum scissors" scheme is presented, which is capable of teleporting the first n Fock-components of arbitrary input states. The operation of the device is analyzed by the application of the $SU(2)$ symmetry of two-mode fields and beam-splitters. A section is devoted to outline an $SU(3)$ theory of passive, lossless optical six-ports (tritters).

Chapter 1

Elements of quantum mechanics and quantum optics

In this chapter I give a brief summary of some elements of quantum mechanics and quantum optics used in the forthcoming chapters. Some parts may seem elementary, and can be found in standard quantum mechanics textbooks [28]. However, as this thesis treats phenomena arising from very fundamental issues, it is indeed worthwhile to go through them briefly. I adopt a somewhat less conventional point of view of quantum mechanics, with more emphasis on multipartite systems. I also intend to outline the exact mathematical structure behind the concepts which are of “everyday use”. The quantum optics part is intended to summarize the basic concepts that are behind my quantum optical results.

1.1 Quantum kinematics

Quantum kinematics describes the mathematical model of *states* of single and multipartite physical systems.

1.1.1 Pure states of systems

Though the concept of a physical system is very fundamental, especially from a philosophical point of view, its definition seems to be somewhat obscure in quantum mechanics. We can however define the mathematical model of *pure states of a physical system* exactly:

POSTULATE 1 *Pure states of a given physical system are unit rays of a Hilbert space $\mathscr{H}$ corresponding to the system.*

Unit rays are equivalence classes of unit-norm Hilbert-space vectors, which are equal up to a complex factor of unit absolute value. It is quite usual however, that a state is represented by an arbitrarily chosen vector from the given unit ray. Vectors of the Hilbert space are denoted by $|\Psi\rangle$, where the letter Ψ specifies the vector, and the notation $|\ldots\rangle$ is called a *ket*:

$$|\Psi\rangle \in \mathscr{H}. \tag{1.1}$$

Moreover, the word "state", if not otherwise stated, stands as a shorthand for "pure state". Occasionally states are represented by unnormalized vectors ("unnormalized states"), these have to be normalized in order to obtain a vector representing the real state they describe.

Roughly one might say that a physical system is an entity the states of which are to be found in the same Hilbert space. Note that the word "same" is not a synonym of "equal" here, $\mathbb{C}^2$ endowed with Hilbert space structure for instance may be a model of an electron spin, or for a photon polarization, which are physically different.

The Hilbert space $\mathscr{H}$ containing the states of the physical system may be either finite or infinite dimensional. In the case of continuous dimensional spaces it is important to require $\mathscr{H}$ to be separable, that is, it has to contain a countable dimensional dense subspace.

The Hilbert space scalar product of vectors $|\Psi_1\rangle$ and $|\Psi_2\rangle$ is denoted by $\langle\Psi_1|\Psi_2\rangle$. The notation $\langle\ldots|$ is called *bra*. $\langle\Psi_1|$ denotes an element of the topological dual space $\mathscr{H}^*$ of $\mathscr{H}$ corresponding to $|\Psi_1\rangle$. In case of finite dimensional Hilbert spaces there is trivially a one-to-one correspondence between bras and kets, while in the continuous variable case, the one-to-one correspondence is a consequence of the Riesz-Fisher representation theorem.

1.1.2 Multipartite systems

Consider two physical systems 1 and 2, the states of which constitute $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively. The question naturally arises: how to describe the states of the system consisting of 1 and 2 as subsystems. The answer is as follows.

POSTULATE 2 *The Hilbert space $\mathscr{H}_{12}$ modeling the states of a bipartite system consisting of subsystems with states in Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ is the tensor product space $\mathscr{H}_{12} = \mathscr{H}_1 \otimes \mathscr{H}_2$.*

Due to the extreme importance of tensor product spaces, let us recall their definition, first for linear spaces:

DEFINITION 1 *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be linear spaces. The tensor product of $\mathscr{H}_1$ and $\mathscr{H}_2$ is a pair $(\mathscr{H}_1 \otimes \mathscr{H}_2, b)$, where $\mathscr{H}_1 \otimes \mathscr{H}_2$ is a linear space, and b is a $\mathscr{H}_1 \times \mathscr{H}_2 \rightarrow \mathscr{H}_1 \otimes \mathscr{H}_2$ bilinear map such that for all vector spaces V and all $R : \mathscr{H}_1 \times \mathscr{H}_2 \to V$ bilinear mappings there exists a unique $L : \mathscr{H}_1 \otimes \mathscr{H}_2 \to V$ linear map such that*

$$R = L \circ b. \tag{1.2}$$

The tensor product is usually denoted by the set $\mathscr{H}_1 \otimes \mathscr{H}_2$ itself. The construction works as follows. One takes an orthonormal basis (ONB) $\{|e_i\rangle\}$ on $\mathscr{H}_1$ and $\{|f_j\rangle\}$ on $\mathscr{H}_2$, where the indices go on the appropriate sets. Then we take

$$\mathscr{H}_1 \otimes \mathscr{H}_2 = \text{span}\{|e_i\rangle \otimes |f_j\rangle\}, \tag{1.3}$$

where the $\otimes$ between $|e_i\rangle$ and $|f_j\rangle$ stands for a "formal product": an ordered pair, where the set of ordered pairs is endowed with vector space structure. In order to obtain Hilbert spaces, we define the scalar product on the product space so that

$$\left(\forall i, j, i', j'\right) (\langle e_i| \otimes \langle f_j|)(|e_{i'}\rangle \otimes |f_{j'}\rangle) = \delta_{i,i'}\delta_{j,j'}. \tag{1.4}$$

The product state basis is then an orthonormal basis. (In continuous variable case, Dirac-deltas appear.) If we take now an arbitrary bilinear operator R acting on $\mathscr{H}_1 \times \mathscr{H}_2$ to any vector space V, the corresponding L is

$$L(|e_i\rangle \otimes |f_j\rangle) = R(|e_i\rangle, |f_j\rangle). \tag{1.5}$$

A special case of the latter equation is the scalar product itself. We can see now that the tensor product space is the set of all linear combinations of the "formal products" of the orthonormal basis elements of the two spaces multiplied, endowed with the appropriate Hilbert space structure.

An important property of bipartite systems is *entanglement*:

DEFINITION 2 *Consider a bipartite system with subsystem 1 and 2. A state vector* $|\Psi_{12}\rangle \in \mathscr{H}_1 \otimes \mathscr{H}_2$ *is separable, if it can be written in a product form:*

$$\left(\exists\, |\Psi_1\rangle \in \mathscr{H}_1,\ |\Psi_2\rangle \in \mathscr{H}_2\right) \quad |\Psi_{12}\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle\,. \tag{1.6}$$

If the state is not separable, it is entangled.

In the case of multipartite systems with more than two subsystems, the definition of the product Hilbert space can be constructed two ways. One may either tensor-multiply subsystems pairwise, thereby producing smaller number of subsystems, and doing so successively, finally a Hilbert space for the whole system is obtained. It can be easily shown that the tensor product is "associative", e.g.

$$(\mathscr{H}_1 \otimes \mathscr{H}_2) \otimes \mathscr{H}_3 = \mathscr{H}_1 \otimes (\mathscr{H}_2 \otimes \mathscr{H}_3), \tag{1.7}$$

therefore the obtained multiple-product Hilbert space is uniquely defined. Alternatively one may directly define an n-fold tensor-product similarly to the tensor product of two Hilbert spaces, where n-linear mappings will be involved. It can also be shown that the two ways are equivalent. Separability can be defined in the same manner as in the case of bipartite states.

It is also important to define the tensor product of two linear operators.

DEFINITION 3 *Let* U *and* V *be a* $\mathscr{H}_1 \to \mathscr{H}_1$ *and a* $\mathscr{H}_2 \to \mathscr{H}_2$ *linear operator respectively. The product operator* $U \otimes V$ *is a* $\mathscr{H}_1 \otimes \mathscr{H}_2 \to \mathscr{H}_1 \otimes \mathscr{H}_2$ *linear operator defined so that*

$$(\forall\, |\Psi_1\rangle \in \mathscr{H}_1)(\forall\, |\Psi_2\rangle \in \mathscr{H}_2)\ (U \otimes V)(|\Psi_1\rangle \otimes |\Psi_2\rangle) = (U\,|\Psi_1\rangle) \otimes (V\,|\Psi_2\rangle). \tag{1.8}$$

1.1.3 Density operators

The most general states of a quantum system are not the pure states described by vectors (unit rays) of the Hilbert space. One has to, for reasons that will become clear in section

1.2, introduce a more general model of states.

DEFINITION 4 *The density matrix for a pure state* $|\Psi\rangle$ *is the projector*

$$|\Psi\rangle\langle\Psi|. \tag{1.9}$$

(The projector $|\Psi\rangle\langle\Psi|$ is a $\mathscr{H}\to\mathscr{H}$ linear operator acting on a vector $|\Phi\rangle$ as

$$(|\Psi\rangle\langle\Psi|)|\Phi\rangle = (\langle\Psi|\Phi\rangle)|\Psi\rangle. \tag{1.10}$$

Generally, a linear operator P is a projector if $P^2 = P$.)

Having a set of states $\{|\Psi_i\rangle\}$, one can take their convex combinations with coefficients $\{p_i|p_i\in[0,1]\},\ \sum_i p_i = 1$

$$\rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|, \tag{1.11}$$

which is a positive semidefinite Hermitian operator of unit trace. Conversely, given a positive semidefinite Hermitian operator of unit trace, one can always find at least one set of states $\{|\Psi_i\rangle\}$, and one set of numbers $\{p_i|p_i\in[0,1]\},\ \sum_i p_i = 1$, such that Eq. (1.11) holds.

DEFINITION 5 *A positive semidefinite* $\mathscr{H}\to\mathscr{H}$ *Hermitian operator* ρ *of unit trace is called a density operator. A density operator describes the most general state of the physical system corresponding to* $\mathscr{H}$*. If* ρ *is a projector, we speak of a pure state, otherwise we speak of a mixed state of the system. A set of states* $\{|\Psi_i\rangle\}$*, and a set of numbers* $\{p_i|p_i\in[0,1]\},\ \sum_i p_i = 1$ *are called a realization of* ρ*, if Eq. (1.11) holds, i.e.* ρ *is the convex combination of the given pure states with the given coefficients.*

1.1.4 Multipartite mixed states and their separability

One may consider multipartite systems in mixed states, for instance a bipartite system with subsystems of Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$. First suppose that system 1 is in a state $|\Psi_1\rangle\in\mathscr{H}_1$, while system 2 is in a state $|\Psi_2\rangle\in\mathscr{H}_2$. The density matrix for the whole system is the tensor product

$$\rho_{12} = |\Psi_1\rangle\langle\Psi_1|\otimes|\Psi_2\rangle\langle\Psi_2|. \tag{1.12}$$

As the density operators for both subsystems are of unit trace,

$$|\Psi_1\rangle\langle\Psi_1| = \mathrm{Tr}_2\,\rho_{12}, \quad |\Psi_2\rangle\langle\Psi_2| = \mathrm{Tr}_1\,\rho_{12}, \tag{1.13}$$

were Tr_1 and Tr_2 denote the *partial trace* on subsystem 1 and 2, on finite dimensional Hilbert spaces:

$$\begin{aligned} \mathrm{Tr}_1\,\rho_{12} &= \sum_{k=0}^{\dim\mathscr{H}_1-1} \langle e_k|\,\rho_{12}\,|e_k\rangle\,, \\ \mathrm{Tr}_2\,\rho_{12} &= \sum_{k=0}^{\dim\mathscr{H}_2-1} \langle f_k|\,\rho_{12}\,|f_k\rangle\,, \end{aligned} \tag{1.14}$$

were $|e_k\rangle$ and $|f_k\rangle$ constitute orthonormal bases on $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively. If we have two independent systems in mixed states ρ_1 and ρ_2, the density matrix of the joint system is

$$\rho_{12} = \rho_1 \otimes \rho_2 \tag{1.15}$$

In general, if the whole system is in the state ρ_{12}, then the state of the subsystems is

$$\rho_1 = \mathrm{Tr}_2\,\rho_{12}, \quad \rho_2 = \mathrm{Tr}_1\,\rho_{12}. \tag{1.16}$$

Eq. (1.16) is the way of extracting the density matrix of a subsystem from a joint density matrix: one must trace over the remaining subsystems. The reason for that is explained later by Eq. (1.25)

But bipartite states are not necessarily of the form in Eq. (1.12) or Eq. (1.15). If the bipartite system was in a pure state, but the state vector was not a product, then the density matrices of the subsystems will describe mixed states. A general state of a bipartite system is described by an arbitrary density operator $\rho_{12} : \mathscr{H}_1 \otimes \mathscr{H}_2 \to \mathscr{H}_1 \otimes \mathscr{H}_2$. The knowledge of the subsystems does not involve all the information on the joint system.

Separability of mixed states is defined as follows:

DEFINITION 6 *A density matrix of a bipartite system is separable if it can be written as a convex combination of projectors describing separable pure states. If a density matrix is not separable, it is inseparable or entangled.*

Notice that separability is not the synonym of the product form of Eq. (1.15). The separability of mixed states is a rather sophisticated issue, with numerous open questions.

Inseparability is a purely kinematical issue. Two subsystems – which may be even spacelikely separated in spacetime – can exist in an inseparable state, and in this case the two subsystems show correlations. However, this is not necessarily surprising, as these correlations may also be classically interpretable. However, if the joint state is pure and inseparable, one necessarily obtains "weird" quantum-mechanical correlations, which will be described in section 1.2.3.

1.2 Quantum dynamics

This section is devoted to the description of interactions and time evolution of quantum systems.

1.2.1 Unitary evolution, canonical quantization

Consider a system in a pure state $|\Psi\rangle$. Suppose that the system is closed, i.e. it does not interact with "the rest of the world".

POSTULATE 3 *A closed quantum system evolves in time as*

$$|\Psi(t)\rangle = \hat{U}(t,t_0)\,|\Psi(t_0)\rangle\,, \text{ where } \hat{U}(t,t_0) = \exp\left(-\frac{i}{\hbar}\hat{H}(t-t_0)\right), \qquad (1.17)$$

$\hat{U}(t,t_0)$ *is a unitary operator, and* t *stands for time.*

Time t appears as a scalar parameter, as quantum mechanics is a non-relativistic theory. The time evolution operators form a one-parameter Lie-group, with the additive parameter t:

$$\hat{U}(t_3,t_2)\hat{U}(t_2,t_1) = \hat{U}(t_3,t_1). \qquad (1.18)$$

As a consequence of their definition, mixed states evolve as

$$\rho(t) = \hat{U}(t,t_0)\rho\hat{U}(t,t_0)^\dagger. \qquad (1.19)$$

In quantum informatics the concept of time is usually not utilized, and the term "unitary evolution" is used in the sense of action of any unitary operator $\hat{U}$. As a consequence of the unitary nature of $\hat{U}$, the infinitesimal generator of time evolution $\hat{H}$ is a Hermitian operator called the Hamiltonian.

Quantum mechanics gives us a recipe for defining the Hamiltonian from the classical description of the system:

POSTULATE 4 *(Rule of Canonical Quantization.) Let there be a physical system described by the Hamiltonian function* $H(q_0, q_1, \ldots, p_0, p_1, \ldots)$, *where q-s and p-s are the canonically conjugate generalized coordinates and momenta. The quantum mechanical Hamiltonian is obtained as follows: take the Hilbert space* $\mathscr{H}$ *of the states of the physical system. Define Hermitian operators* $\hat{q}_i$, $\hat{p}_i$ *on a dense subspace of* $\mathscr{H}$ *so that for their commutators*

$$[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{i,j} \tag{1.20}$$

holds. The Hamiltonian describing the system is then

$$\hat{H} = H(\hat{q}_0, \hat{q}_1 \ldots, \hat{p}_0, \hat{p}_1, \ldots). \tag{1.21}$$

It is important to note that Eq. (1.20) can be satisfied neither on a finite dimensional Hilbert space, nor with linear operators defined on the whole $\mathscr{H}$ in the infinite dimensional case. In finite dimensional case for instance, the commutator of position and momentum is strictly off-diagonal in the position basis. There is still a way to define “quasi” canonically conjugate “position” and “momentum” operators on finite dimensional Hilbert spaces, which will be discussed is section 3.4.

We also remark that there are physical quantities, such as spin, which do not have a classical counterpart at all. For these, the Hamiltonian is constructed phenomenologically, so that the predictions of the model are consistent with the experimental results.

1.2.2 Quantum measurement

The measurable quantities are modeled by $\mathscr{H} \to \mathscr{H}$ Hermitian operators in quantum mechanics. Hermitian operators are also referred to as *observable operators*. We have already encountered some examples in the last chapter: generalized coordinates and momenta, and the Hamiltonian, which is the operator of energy. If a quantity in argument has a classical counterpart, that is, it is a function $M(q_0, q_1, \ldots, p_0, p_1, \ldots)$ of generalized coordinates and momenta, its operator has to be built up according to the canonical quantization rule: $\hat{M}(\hat{q}_0, \hat{q}_1, \ldots, \hat{p}_0, \hat{p}_1, \ldots)$. Otherwise, any Hermitian operator describes an observable.

The possibly most troublesome postulate of quantum mechanics is the one connecting the quantum mechanical model of measurement with real experiments:

POSTULATE 5 *(von Neumann measurement) Suppose that we have a measuring apparatus that measures the quantity M, and let $\hat{M}$ be the corresponding observable operator. Let the system which is subjected to a measurement be in the pure state $|\Psi\rangle$ at the instant before the measurement. Let $(|m_i\rangle, m_i)$ be the set of eigenvectors (eigenstates) and corresponding eigenvalues of $\hat{M}$. The measurement result is one of the eigenvalues, and the i-th eigenvalue is obtained with probability $p_i = |\langle m_i|\Psi\rangle|^2$. (In case of continuous spectrum, $|\langle m_i|\Psi\rangle|^2$ describes a probability density.) At the instant of the measurement, the state of the system becomes the corresponding eigenstate $|m_i\rangle$ (von Neumann projection principle).*

Hermitian operators have real eigenvalues, thus the result displayed by the pointer of the measurement apparatus is a real number. The eigenvectors form an ONB on $\mathscr{H}$, thus the events "the i-th eigenvalue has been measured" form a complete set of events due to the normalization of the state vectors. Vectors in the same unit ray provide the same measurement results with the same probabilities.

In the case of continuous dimensional Hilbert spaces, $\hat{M}|m\rangle = m|m\rangle$, and $f(m) = \langle m|\Psi\rangle$ is usually called the *wavefunction*, especially if the observable under consideration is a generalized coordinate. $|f(m)|^2$ gives the probability density for the measurement of $\hat{M}$.

The expectation (or mean) value of the observable $\hat{M}$ is

$$\langle\hat{M}\rangle = \langle\Psi|\hat{M}|\Psi\rangle\,. \tag{1.22}$$

If the system is in a mixed state ρ, the expectation value reads

$$\langle\hat{M}\rangle = \mathrm{Tr}(\rho\hat{M}). \tag{1.23}$$

Taking a realization of ρ, this reads

$$\langle\hat{M}\rangle = \sum_i p_i \langle\Psi_i|\hat{M}|\Psi_i\rangle\,. \tag{1.24}$$

It is clear that the p_i coefficients are probabilities of having the system described by the mixed state ρ in the state $|\Psi_i\rangle$, thus mixed states can model an external source of stochasticity (e. g. a random number generator), or the lack of knowledge on the quantum state.

The projection principle can be formulated mathematically as follows. Let $\hat{P}_i = |m_i\rangle\langle m_i|$ be the projector for the actual measurement result. The state of the system after the measurement is the normalized version of $\hat{P}_i|\Psi\rangle$ if it was in the state $|\Psi\rangle$, while it is proportional to $\hat{P}_i\rho\hat{P}_i$, if it was in the state ρ.

In the case of a multipartite system, one may carry out a measurement on a given subsystem, e.g. subsystem 1. This results in the projection $\hat{P}_i \otimes \hat{1}$, where $\hat{P}_i$ corresponds to an observable on subsystem 1, and $\hat{1}$ stands for the identity operator of the rest of the system. The partial traces in Eq. (1.16) yielding the density operator for a subsystem of a bipartite system can be easily derived from the condition, that for an observable $\hat{M}_A$ on system A,

$$\langle \hat{M}_A \rangle = {}_{AB}\langle \hat{M}_a \otimes \hat{1}_B \rangle_{AB} \tag{1.25}$$

should hold for any state of the bipartite state. Thus partial trace is a kind of averaging in the subsystem we are not interested in.

We have tacitly assumed that our measurement apparatus can make a difference between all possible outcomes. In reality, some information may be lost due to the technical construction of the device. An example for this is photodetection in the laboratory, where no detectors can tell the exact number of photons at the present state of art. Another possibility is that our measurement is made on a larger system, though we intended to measure on a subsystem only. These facts suggest that a more general model for a measurement would be more appropriate in some cases. This leads to the concept of generalized (POVM) measurements. POVM means “Positive Operator Valued Measure”, a mathematical object describing these kind of measurements. Concerning them, I refer to the literature [63, 64].

The instantenous change to the state imparted by quantum measurement is a very counterintuitive feature of quantum mechanics, and most of the paradoxes are deeply related to this. I adopt the von Neumann principle as a given fact. It is far beyond the scope of this dissertation even to summarize the questions about it. But my results may be helpful to know one's way about quantum measurement.

1.2.3 Nonlocality

Though in this thesis I will not investigate nonlocality directly, it is so central in the physics of multipartite quantum systems, and so deeply related to quantum teleportation, that our introduction would be incomplete without making some comments on it. The concept of quantum nonlocality cannot be fully understood in the framework of standard quantum mechanics. It is quite natural that there are statistical correlations between the behaviors of subsystems of e.g. a bipartite system. Some of these are not surprising, as they can be interpreted classically. The attention to *really* paradoxical correlations between quantum systems of two well-separated subsystems was first drawn by Einstein, Podolsky and Rosen in their classic 1935 paper [31]. The word "locality" in a more physical sense appears in the context of hidden variable theories suggested by D. Bohm [10, 11], which were completely successful in describing single-particle quantum mechanics. The key idea was to supplement the state vector $|\Psi\rangle$ with some additional parameters, which are supposed to be unmeasurable and uncontrollable (hence the term "hidden"), and together with them, a completely deterministic description of the physical system can be achieved. Randomness enters through the random initial conditions of hidden parameters.

J. S. Bell, who was filled with enthusiasm about Bohm's theories, considered the possibility of describing multipartite systems. Considering certain correlations of outcomes of measurements carried out on two subsystems, he found inequalities – later named after him –, which should hold, provided that a local hidden variable theory is valid [5]. Locality means that the result of the measurement on one apparatus should depend only on the setting of that apparatus, and not on the other. As the description is now deterministic, locality is a strong restriction. Bell found that quantum mechanical predictions for systems in entangled pure states contradict with the inequalities. Thus one can conclude that no local hidden variable theory can describe predictions of quantum mechanics.

The first experiment to verify Bell inequalities was proposed by Clauser, Horne, Shimony, and Holt [27]. Recently, as entangled states became available experimentally, a lot of work has been done in order to generalize Bell inequalities [46].

A state that violates Bell's inequalities is referred to as *nonlocal*. In the case of pure states, all inseparable states are nonlocal, and vice versa. In the case of mixed states, the picture is more Byzantine: there exist states for instance, which are inseparable, but do not violate Bell's inequalities. Some of these states for instance constitute the class of the so-called Werner sates [78].

1.3 Elements of quantum optics

In order to study fundamental features of quantum mechanics in actual experimental situations, one must find a sufficiently simple and well controllable physical system, which exhibits them in a pure form. Light appears to be an excellent subject for such applications. In this section I review some elements of quantum optics, the quantum mechanical description of light.

1.3.1 Monochromatic modes

Light is described by the Maxwell equations and appropriate boundary conditions in classical physics [40]. In vacuum, in the absence of free charges and currents, they read

$$\begin{aligned} \nabla\mathbf{E}(\mathbf{r},t) &= 0, \\ \nabla\mathbf{B}(\mathbf{r},t) &= 0, \\ \nabla\times\mathbf{E}(\mathbf{r},t) &= -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t), \\ \nabla\times\mathbf{B}(\mathbf{r},t) &= \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t), \end{aligned} \tag{1.26}$$

where $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ stand for the electric and magnetic field vector as a function of position and time respectively, and c is the speed of light in vacuum. We use CGS units. The first pair of equations enables us to introduce a vector potential from which the field quantities are obtained as

$$\mathbf{E}(\mathbf{r},t) = -\frac{1}{c}\frac{\partial}{\partial t}A(\mathbf{r},t), \qquad \mathbf{B}(\mathbf{r},t) = \nabla\times A(\mathbf{r},t). \tag{1.27}$$

The vector potential can be introduced in many ways, we prescribe the gauge condition

$$\nabla A(\mathbf{r},t) = 0, \tag{1.28}$$

that is, we work in the Coulomb gauge.

The second pair of the Maxwell equations in Eq (1.26) imply that the wave equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) A(\mathbf{r},t) = 0 \tag{1.29}$$

should be satisfied. The general solution can be written as

$$A(\mathbf{r},t) = \sum_l \left(\mathbf{u}_l(\mathbf{r})\alpha_l e^{-i\omega_l t} + c.c.\right), \tag{1.30}$$

where the summation is carried out over a complete set of modes characterized by the orthonormal set of mode functions $\mathbf{u}_l(\mathbf{r})$ satisfying the Helmholtz equation

$$\left(\nabla^2 + \frac{\omega_l^2}{c^2}\right)\mathbf{u}_l(\mathbf{r}) = 0 \tag{1.31}$$

with appropriate boundary conditions. The different solutions are the *modes*, are indexed by the mode index l, and α_l and ω_l are the amplitude and the frequency of the given mode. The set of boundary conditions that determine the solutions of the Helmholtz equations represent the geometry of the actual physical scenario under consideration. The mode functions may be for instance Gaussian functions in case of having a good laser cavity. *There is always a solved classical optical problem behind field quantization.* Notice that possessing a set of mode functions, we have divided the electromagnetic field into a set of harmonic oscillators described by $e^{-i\omega_l t}\alpha_l$, corresponding to monochromatic modes.

Treating traveling wave fields, it is usual to impose periodic boundary conditions by considering a large cubic volume L^3. In this case, the mode functions are plane waves:

$$\mathbf{u}_{\mathbf{k},s}(\mathbf{r}) = \frac{1}{L^{3/2}}\varepsilon_{\mathbf{k},s}e^{i\mathbf{kr}}, \qquad \omega_{\mathbf{k},s} = c\sqrt{\mathbf{k}^2}. \tag{1.32}$$

The role of the mode index l of Eq. (1.30) is now played by the pair $(\mathbf{k},s)$. The vector index $\mathbf{k}$ can take the values

$$k_i = \frac{2\pi}{L}\begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}, \tag{1.33}$$

where all three n-s are integers, and $\varepsilon_{\mathbf{k},s}$-s are the polarization vectors with the properties $s \in \{1,2\}$, $\varepsilon_{\mathbf{k},s}\varepsilon_{\mathbf{k},s'} = \delta_{s,s'}$, and $\varepsilon_{\mathbf{k},1} \times \varepsilon_{\mathbf{k},2} = \frac{\mathbf{k}}{|\mathbf{k}|}$. The larger the volume L, the more modes appear. To be exact, one should take infinite quantization volume, obtaining a continuum of modes. In situations investigated in quantum optics however, it is *assumed* that only a single mode, or a few distinct modes are excited. This mere assumption is well justified by experiments. We remark here that pulsed lasers are non-monochromatic, but, as it will be discussed in section 1.3.4, non-monochromatic fields of certain coherence properties can be treated as single mode fields.

1.3.2 Quantization of fields, harmonic oscillators

In the last subsection we have seen how the electromagnetic field can be split into modes, each mode described by a harmonic oscillator. The quantization of the field resides in quantization of these oscillators.

We define the *quadratures* as

$$q_l = \frac{1}{c}\sqrt{\frac{\omega_l}{\hbar}}(\alpha_l + \alpha_l^*), \qquad p_l = -i\frac{1}{c}\sqrt{\frac{\omega_l}{\hbar}}(\alpha_l - \alpha_l^*), \tag{1.34}$$

which are proportional to canonically conjugate quantities, the “position” and “momentum” of the actual oscillator. Optically, they describe the cosinusoidally and sinusoidally varying part of the wave. Then we apply the rule of canonical quantization described in postulate 4: we define Hermitian operators $\hat{q}_l$ and $\hat{p}_l$ so that

$$[\hat{q}_l, \hat{p}_{l'}] = i\delta_{l,l'}. \tag{1.35}$$

(Note, that $\hbar$ does not appear on the right hand side, as the quadratures are definied to be dimensionless.) We now have quantum oscillators for each mode. It is also worth defining the following non-Hermitian operators:

$$\hat{a}_l = \frac{\hat{q}_l + i\hat{p}_l}{\sqrt{2}}, \tag{1.36}$$

for which

$$[\hat{a}_l, \hat{a}_{l'}^\dagger] = \delta_{l,l'}. \tag{1.37}$$

These are called *annihilation* and *creation* operators for a reason that will become clear later. Up to an ω_l-dependent factor, they are quantized versions of the α_l amplitudes and their complex conjugates, respectively. Namely, the quantized vector potential (c.f. Eq. (1.30)) is then represented by the operator.

$$\hat{\mathbf{A}}(\mathbf{r},t) = c\sum_l \sqrt{\frac{\hbar}{2\omega_l}}\left(\mathbf{u}_l(\mathbf{r})\hat{a}_l e^{-i\omega_l t} + h.c.\right). \tag{1.38}$$

Quantum operators of quadratures can be expressed by the creation and annihilation operators as

$$\hat{q}_l = \frac{\hat{a}_l + \hat{a}_l^\dagger}{\sqrt{2}}, \qquad \hat{p}_l = \frac{\hat{a}_l - \hat{a}_l^\dagger}{\sqrt{2}i}. \tag{1.39}$$

In case of having plane waves as mode functions, the mode index $\hat{l}$ has to be replaced by the pair $\mathbf{k}, s$. Thus Eq. (1.38) for instance reads

$$\hat{\mathbf{A}}(\mathbf{r},t) = c\sqrt{\frac{\hbar}{2L^3}} \sum_{k,s} \frac{1}{\sqrt{\omega_\mathbf{k}}} \left(\varepsilon_{\mathbf{k},s} \hat{a}_{\mathbf{k},s} e^{i(\mathbf{kr}-\omega_\mathbf{k} t)} + h.c. \right), \tag{1.40}$$

c. f. Eq. (1.32). (Let me remark that in Eqs. (1.38) and (1.40), the left hand side is in the Heisenberg picture of quantum mechanics, that is, the time development is included in the operator, while the $\hat{a}$ and $\hat{a}^\dagger$ operators on the right hand side are time-independent ones in the Schrödinger picture.)

The Hermitian operators $\hat{n}_l = \hat{a}_l^\dagger \hat{a}_l$ is related to the energy of the oscillator representing the given mode. Considering a single mode only, the spectrum of $\hat{n}$ is bounded from below, and the eigenstates can be labelled by natural numbers:

$$\hat{n} |n\rangle = n |n\rangle, \quad n \in \mathbb{N}. \tag{1.41}$$

These states are called photon number states or Fock states [33], and they form a complete basis on the Hilbert space of the given mode. The Hilbert space of the entire electromagnetic field is the tensor product of the oscillator Hilbert spaces. Physically, n is interpreted as the number of photons in the given mode, and $|n\rangle$ is a state with exactly n photons. The state $|0\rangle$ is called *vacuum*, it describes the lack of any photons. The operators $\hat{a}^\dagger$ and $\hat{a}$ are photon creation and annihilation operators:

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \qquad \hat{a} |n\rangle = \sqrt{n} |n-1\rangle, \tag{1.42}$$

describing absorption and emission of a photon in the given mode.

1.3.3 Photodetection

According to Postulate 5 an ideal photodetector, which measures the state of a given mode should work as follows: suppose that the field to be measured is in state $|\Psi\rangle$. Provided that the detector measures n photons the state of the field should be projected to $|n\rangle$, and this event should occur with probability $|\langle n|\Psi\rangle|^2$. Throughout this thesis I will adopt this rather simple picture. The aim of the following paragraphs is to relate this naïve and idealized approach to reality.

For simplicity, let us assume that we have completely polarized light, thus the polarization index s can be omitted. Following Glauber [36, 70], we introduce the positive and negative parts of the electric field, which is obtained from Eq. (1.40) by the application of Eqs. (1.27,1.36):

$$\hat{\mathbf{E}}^{(+)}(\mathbf{r},t) = i\sum_l \sqrt{\frac{1}{2}\hbar\omega_l}\mathbf{u}_l(\mathbf{r})e^{-i\omega_l t}\hat{a}_l,$$
$$\hat{\mathbf{E}}^{(-)}(\mathbf{r},t) = -i\sum_l \sqrt{\frac{1}{2}\hbar\omega_l}\mathbf{u}_l^*(\mathbf{r})e^{i\omega_l t}\hat{a}_l^\dagger, \tag{1.43}$$

which are Hermitian conjugate to each other, and the electric field at spacetime point $\mathbf{r},t$ can be expressed as

$$\hat{\mathbf{E}}(\mathbf{r},t) = \hat{\mathbf{E}}^{(+)}(\mathbf{r},t) + \hat{\mathbf{E}}^{(-)}(\mathbf{r},t). \tag{1.44}$$

A more realistic model of a detector is a very small object at point $\mathbf{r}$, with frequency-independent photon absorption property. The detection event is the absorption of a single photon. If the field was in the state $|\Psi_i\rangle$, and after the detection it ends up in the state $|\Psi_f\rangle$, the field matrix element for this event is

$$\langle\Psi_f|\,\hat{\mathbf{E}}^{(+)}(\mathbf{r},t)\,|\Psi_i\rangle\,. \tag{1.45}$$

In this picture of detection however, we don't know the actual final state, thus in order to obtain the transition rate per unit time, we have to sum up the square absolute values of transition matrix elements for a complete set of $|\Psi_f\rangle$. By supposing that the detector is of broad bandwidth and weights the contributions of the perturbation equally, one obtains for the counting rate, i.e. detection probability per unit time

$$P_{\text{det}}(\mathbf{r},t) = \langle\Psi_i|\,\hat{\mathbf{E}}^{(-)}(\mathbf{r},t)\hat{\mathbf{E}}^{(+)}(\mathbf{r},t)\,|\Psi_i\rangle\,. \tag{1.46}$$

In the case of a single-mode field, the counting rate is proportional to the mean photon number $\langle\hat{n}\rangle$, as one would expect. Since the measurement does not determine the final state exactly, we are left with a density matrix. A real detector does not realize a von Neumann measurement, it can only be described as a generalized (POVM) measurement.

Thus in reality, detectors measure intensity, i.e. mean photon number. How could one construct a detector which really "counts" photons? There are detectors of single photon *sensitivity*, that is, they fire even if only a single photon is present. Notice that they may realize projective measurement to the vacuum state very well: if we can assure that at most a single photon is present in any case, according to Eq. (1.46), they appear to project onto $|0\rangle$ or $|1\rangle$. This can be exploited in order to *approximately* realize photon counting: guiding the mode to be measured to a sufficiently large multiport, it is very likely, that at most one photon is present in any of the output modes at the same time. Putting detectors of single photon sensitivity in front of the outputs of the multiport, photon counting is realized approximately, though unfortunately the contributions due to the presence of more than one photons in the output ports cannot be neglected. See Refs. [60, 49] for details.

Another possibility is to exploit two-photon absorption, the second order effect that was not taken into account in the above description of Glauber's photodetection theory. An analysis can be found in [75].

We can conclude that, – though at the present state of art photon counting is not really feasible –, it is not impossible theoretically, and thus it is worth investigating situations from the point of view of von Neumann's projection principle in our situation as well. On the other hand, situations which are feasible experimentally can be analyzed only if one understands the behavior of the system under idealized conditions.

1.3.4 Non-monochromatic fields, coherence

Most of the experimental schemes treated in chapter 5 utilize short laser pulses. Therefore it is necessary to mention, how modes can be defined in that case. This problem is connected with the concept of coherence.

Suppose that we have two photodetectors at spacetime points $\mathbf{r}_1, t_1$ and $\mathbf{r}_2, t_2$. According to Glauber's photodetection theory [36] the first order correlation of detection events is the expectation value

$$G^{(1)}_{\mu,\nu}(\mathbf{r}_1,t_1,\mathbf{r}_2,t_2) = \mathrm{Tr}\left(\rho \mathbf{E}^{(-)}_{\mu}(\mathbf{r}_1,t_1)\mathbf{E}^{(+)}_{\nu}(\mathbf{r}_2,t_2)\right), \tag{1.47}$$

where the indices μ, ν stand for the vector components of the field. The field is said to possess first order coherence if this correlation function factors as

$$G^{(1)}_{\mu,\nu}(\mathbf{r}_1,t_1,\mathbf{r}_2,t_2) = \mathscr{E}^*_\mu(\mathbf{r}_1,t_1)\mathscr{E}_\nu(\mathbf{r}_2,t_2), \tag{1.48}$$

where $\mathscr{E}$ is a complex solution of the wave equation. Monochromatic and completely polarized fields are for instance coherent to the first order. It was shown by Titulauer and Glauber [70] that if a polychromatic completely polarized field of a single spatial mode possesses first order coherence, it can always be treated as if it were a single mode field. One may construct new mode functions from those defined in Eq. (1.30) as

$$\mathbf{v}_\mathbf{l}(\mathbf{r},t) = i\sum_\mathbf{k} \gamma_{\mathbf{l},\mathbf{k}}\sqrt{\frac{1}{2}\omega_\mathbf{k}}\mathbf{u}_\mathbf{k}(\mathbf{r},t)e^{i\omega_\mathbf{k}t}, \tag{1.49}$$

where the coefficients $\gamma_{\mathbf{l},\mathbf{k}}$ constitute a unitary matrix. The new annihilation operators are defined by the canonical transformation

$$\hat{b}_\mathbf{l} = \sum_\mathbf{k} \gamma^*_{\mathbf{l},\mathbf{k}}\hat{a}_\mathbf{k}. \tag{1.50}$$

It is shown in Ref. [70] that the coherence condition in Eq. (1.48) implies that the transformation can be chosen in such a way that only one of the new modes are excited. It is therefore reasonable in case of non-monochromatic fields possessing first order coherence to speak about single mode fields.

1.3.5 The Wigner-function representation

Consider a single-mode electromagnetic field, i.e., a harmonic oscillator. In classical physics, the state of an oscillator is completely described by the statistics of the amplitude α defined in Eq. (1.30). This complex quantity involves two observables, namely the quadratures q and p defined in Eq. (1.39) which are, up to a factor of $\sqrt{2}$, the real and imaginary part of α.

In quantum mechanics on the other hand, it is not possible to measure these quantities simultaneously, according to the Heisenberg uncertainty principle. The question arises whether – in spite of this fact – one can represent a quantum state as some kind of a "probability distribution" defined on the *phase space*, i.e., the complex plane involving all α-s. It turns out that there are infinitely many ways of doing so, and the functions defined this way may lose an important property of probability distributions: positive definitivity. Therefore they are called *quasiprobabilities*. In what follows I summarize the properties of the Wigner function which, to some extent, resembles classical phase-space distributions the most. It is a very brief summary, for details see the excellent texts and summaries available on the topic, e.g. Refs. [52, 65].

Wigner's original definition of the distribution function reads for a field state ρ:

$$W(q,p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, e^{ipx} \left\langle q - \frac{x}{2} | \hat{\rho} | q + \frac{x}{2} \right\rangle dx. \tag{1.51}$$

Alternatively one may define the *symmetrically ordered characteristic function* as

$$\chi_s(\xi) = \mathrm{Tr}(\hat{\rho} e^{\xi \hat{a}^\dagger - \xi^* \hat{a}}), \tag{1.52}$$

and define the Wigner function as a two-dimensional Fourier transform:

$$W(q,p) = \frac{1}{\pi^2} \int d^2\xi\, \chi_s(\xi) e^{\alpha \xi^* - \alpha^* \xi}, \quad \alpha = \frac{q+ip}{\sqrt{2}}. \tag{1.53}$$

The two definitions can be shown to be equivalent. The function defined this way is real-valued, and has certain interesting properties. The most important of these from our point of view are the following:

1. The Wigner function is normalized:
$$\int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp W(q,p) = 1. \tag{1.54}$$

2. The *overlap property*: for *any* two operators $\hat{F}_1$ and $\hat{F}_2$, if we define "Wigner functions" similarly to Eq. (1.51) as
$$W_{F_i}(q,p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, e^{ipx} \left\langle q - \frac{x}{2} \middle| \hat{F}_i \middle| q + \frac{x}{2} \right\rangle dx, \qquad i = 1,2, \tag{1.55}$$
then
$$\mathrm{Tr}(\hat{F}_1 \hat{F}_2) = 2\pi \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp W_{F_1}(q,p) W_{F_2}(q,p) \tag{1.56}$$
holds. In particular, if $\hat{F}_1$ is an observable, and $\hat{F}_2$ is a density operator, then this formula allows the calculation of the expectation value of the observable. Expectation values can be regarded as projections of the Wigner function filtered with the corresponding function of the observable.

3. *Marginals.* Though the Wigner function itself is not in general positive definite, averaging it in either of its variables we obtain positive definite functions:

$$P_q(q) = \int_{-\infty}^{\infty} dp W(q,p),$$
$$P_p(p) = \int_{-\infty}^{\infty} dq W(q,p), \tag{1.57}$$

which are the probability densities of measured values of the observables $\hat{q}$ and $\hat{p}$ respectively. One may define even more general marginals, but these two will be sufficient for my purposes.

Wigner functions can be defined for multipartite systems as well. Similarly to the definitions in Eqs. (1.52) and (1.53), for a bipartite system one may define

$$\chi_s(\xi_1,\xi_2) = \mathrm{Tr}(\hat{\rho} e^{\xi_1 \hat{a}_1^\dagger - \xi_1^* \hat{a}_1} e^{\xi_2 \hat{a}_2^\dagger - \xi_2^* \hat{a}_2}),$$
$$W(q_1,p_1,q_2,p_2) = \frac{1}{\pi^2} \int d^2\xi_1 \int d^2\xi_2 \chi_s(\xi_1,\xi_2) \quad e^{\alpha_1 \xi_1^* - \alpha_1^* \xi_1} e^{\alpha_2 \xi_2^* - \alpha_2^* \xi_2}, \tag{1.58}$$
$$\alpha_1 = \sqrt{2}(q_1 + ip_1), \alpha_2 = \sqrt{2}(q_2 + ip_2).$$

Though for separable states

$$W(q_1,p_1,q_2,p_2) = W(q_1,p_1)W(q_2,p_2) \tag{1.59}$$

holds, this is not true in general. The Wigner function for a subsystem is obtained by averaging over the other subsystem:

$$W(q_1,p_1) = \int_{-\infty}^{\infty} dq_2 \int_{-\infty}^{\infty} dp_2 W(q_1,p_1,q_2,p_2),$$
$$W(q_2,p_2) = \int_{-\infty}^{\infty} dq_1 \int_{-\infty}^{\infty} dp_1 W(q_1,p_1,q_2,p_2). \tag{1.60}$$

1.3.6 Coherent states, coherent-state representations

Consider a single-mode electromagnetic field. A coherent state is an eigenstate of the annihilation operator:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \tag{1.61}$$

The Wigner function of a coherent state is a Gaussian centered at the phase-space point α, with a width corresponding to vacuum noise, showing that this family of states describes a "most classical" state of a quantum oscillator with mean amplitude α.

The Glauber displacement operator

There are several other approaches to the definition of coherent states. One may introduce the *Glauber displacement operator*

$$\hat{D}(\beta) = e^{\beta\hat{a}^\dagger - \beta^*\hat{a}}, \tag{1.62}$$

possessing the remarkable property of "displacing" states in the phase space: if a state ρ is described by the Wigner function $W(\alpha)$, then for the Wigner function $W'(\alpha)$ corresponding to the state $\rho' = \hat{D}(\beta)\rho\hat{D}(\beta)^\dagger$,

$$W'(\alpha) = W(\alpha+\beta) \tag{1.63}$$

holds. Thus coherent states are displaced vacuum states:

$$|\alpha\rangle = \hat{D}(\alpha)\,|0\rangle\,. \tag{1.64}$$

A frequently exploited property of the Glauber displacement operators is

$$\hat{D}(\alpha)\hat{D}(\beta) = e^{\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)}\hat{D}(\alpha+\beta). \tag{1.65}$$

From this and Eq (1.64) follows that a displaced coherent state in general reads

$$\hat{D}(\beta)\,|\alpha\rangle = e^{\frac{1}{2}(\beta\alpha^* - \beta^*\alpha)}\,|\alpha+\beta\rangle\,. \tag{1.66}$$

Coherent-state representations

Coherent states are eigenstates of the non-Hermitian operator $\hat{a}$. They are not mutually orthogonal, namely the scalar product of two coherent states reads

$$\langle\alpha|\beta\rangle = e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)+\alpha^*\beta}. \tag{1.67}$$

The absolute value of this product is

$$|\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2}, \tag{1.68}$$

that is, the farther two coherent states are on the phase space, the closer they are to orthogonality.

On the other hand, the set of coherent states is complete:

$$\frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha\, |\alpha\rangle\, \langle\alpha| = 1. \tag{1.69}$$

This means that coherent states constitute a basis of the Hilbert space of the harmonic oscillator, but this basis is not orthogonal. Moreover, it can be found that this basis is overcomplete: there is an ambiguity in expanding a state on this basis, and even a subset of the coherent states turns out to be sufficient to expand a state. The mathematical background for these selections is the theorem of Cahill [23].

There are representations exploiting the whole phase space. A famous one is Glauber's analytic representation: any pure state $|\Psi\rangle$ of a mode can be written as

$$|\Psi\rangle = \int_{\mathbb{C}} e^{-\frac{|\alpha|^2}{2}} f(\alpha^*) |\alpha\rangle\, d^2\alpha. \tag{1.70}$$

It is the only way of defining a representation such that the function $f(\alpha^*)$ is analytical on the whole complex plane. I will exploit this representation in chapter 4. The generalization to mixed states is the so-called P-representation: any state ρ can be written as

$$\rho = \int_{\mathbb{C}} d^2\alpha P(\alpha)\, |\alpha\rangle\, \langle\alpha|\,. \tag{1.71}$$

The P function, or Glauber function turns out to be another quasiprobability deeply related to the Wigner function.

The question naturally arises whether one can find a basis on which at least a sufficiently large set of states can be written as a superposition of coherent states in a subset of the phase space. An example of such methods, the one-dimensional representations exploit the fact that a set of coherent states corresponding to a one-dimensional curve on the complex phase-space form a complete basis. This makes it possible to express any state as a superposition of coherent states placed along a one-dimensional curve, instead of taking all possible coherent states of the phase-space with nonzero weights [1, 41]. A simple example is when the superposition integral goes over either the real or the imaginary axis [45], which will be applied in chapter 4. Even more can be done in order to decrease the number of coherent states involved. It has been shown that even a finite number of coherent states may be sufficient to approximate certain classes of states surprisingly well [42].

1.3.7 Parametric down-conversion

The entangled states in optical experiments are usually generated by parametric down-conversion. In this nonlinear optical process a photon of a strong incoming coherent light field (pump) decays into two photons with some probability. The decay is due to the elastic interaction of the light with the electron system of the medium which it travels through. This medium is usually a nonlinear optical crystal, and both the wave vector and the polarization of the outcoming down-converted photons is determined by the phase-matching condition, i.e. the conservation of momentum.

In the quantum mechanical treatment the pump is not quantized, and the process is described by the interaction Hamiltonian

$$\hat{H} = \kappa \hat{a}_1 \hat{a}_2 + h.c. \tag{1.72}$$

The factor κ includes pump strength and effective nonlinearity of the medium.

The resulting two-mode state (in the non-degenerate case), if the intensity is sufficiently small, may be well approximated as a superposition of vacuum and the two-mode Fock-state $|1,1\rangle$, though with some probability states with two or more photons are also present. The propagation and polarization direction of the modes is fixed by the conservation of both photon momentum and energy, the so-called *phase matching condition* [82]. Exploiting polarization relations determined by the phase matching, the resulting states are also polarization entangled.

1.3.8 $SU(2)$ theory of beam splitters

A beam splitter, or linear coupler is a device coupling two input and two output ports, each of which are single modes of the electromagnetic field. In this way, it is a basic component of all interferometric schemes. The details of the theory of lossless passive beam splitters can be found in Ref. [24]. Our task is now to emphasize the strictly group-theoretical aspects of the theory. Let $\hat{a}_1$ and $\hat{a}_2$ denote the annihilation operators of the input ports, and $\hat{b}_1$ and $\hat{b}_2$ those of the outputs respectively, as depicted in Fig. 1.1. In a

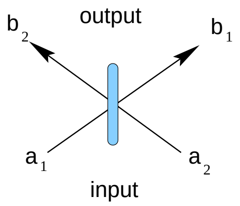

Figure 1.1: A beam splitter

passive lossless linear beam splitter (or in a linear coupler) these are connected via

$$\hat{b}_i = \sum_{j=1}^{2} U_{ij}\hat{a}_j, \quad i = 1,2 \quad U \in SU(2), \tag{1.73}$$

thus the 2×2 matrix U is a matrix corresponding to the fundamental representation of $SU(2)$, most generally:

$$U = \begin{pmatrix} e^{i\varphi_t}\cos\theta & e^{i\varphi_r}\sin\theta \\ -e^{-i\varphi_r}\sin\theta & e^{-i\varphi_t}\cos\theta \end{pmatrix}. \tag{1.74}$$

The three parameters describing the beam splitter correspond to a parametrization of $SU(2)$. They have physical meaning: $\cos(\theta)$ is the beam-splitter transmittance, while φ_t and φ_r are the phase shifts imparted by the beam splitter to the transmitted and reflected beam respectively.

Both the input and the output pair of modes can be regarded as two-dimensional oscillators, and used as bases for representations of $SU(2)$ symmetry. According to the Schwinger representation of angular momenta [9] the generators $\hat{L}_1, \hat{L}_2, \hat{L}_3$ of the $\mathfrak{su}(2)$ Lie-algebra can be constructed as

$$\hat{L}_k = \begin{pmatrix} \hat{a}_1^\dagger & \hat{a}_2^\dagger \end{pmatrix} \begin{pmatrix} \frac{1}{2}\hat{\sigma}_k \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}, \quad k = 1,2,3, \tag{1.75}$$

where the $\hat{\sigma}_k$-s are the Pauli-matrices. The output operators $\hat{b}_i$ realize the $\mathfrak{su}(2)$ Lie-algebra in the same way, these generators will be denoted by $\hat{K}_1, \hat{K}_2, \hat{K}_3$. The consequence of this is that the two-mode number states $|n,m\rangle$ can be divided into $SU(2)$-multiplets. We consider input states, the method is the same for the output states. One may construct the operator

$$\hat{l} = \frac{1}{2}(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2), \tag{1.76}$$

from which the operator of the "square of angular momentum", is the Casimir-operator of the algebra, can be constructed as

$$\hat{L}^2 = \hat{l}(\hat{l} + \hat{1}). \tag{1.77}$$

The multiplets can be indexed by the eigenvalue of the Casimir-operator. In the theory of the angular momenta, it is usual to use the eigenvalue of $\hat{l}$ instead, as it is in a one-to-one correspondence with the eigenvalue of $\hat{L}^2$. In our case, a multiplet of index l is the set of the number states of $2l = n + m$. The states in a multiplet are indexed by the eigenvalues $l_3 = \frac{1}{2}(n - m)$ of $\hat{L}_3$. Thus instead of the photon number, states can be indexed as

$$|n, m\rangle = |l, l_3\rangle. \tag{1.78}$$

The ladder operators $L_+ = \hat{a}_1^\dagger \hat{a}_2$, $L_- = \hat{a}_2^\dagger \hat{a}_1$ defined in the standard way can be applied to increase and decrease the index l_3. The same relabelling of states can be defined for the number states at the output.

The beam splitter itself is also an $SU(2)$ device according to Eq. (1.73). There are two important consequences of this fact. One of these is that the Lie-algebras at the input and at the output are related as

$$\hat{K}_k = \sum_{l=1}^{3} \mathscr{O}_{kl} \hat{L}_l, \quad k = 1, 2, 3, \tag{1.79}$$

where $\mathscr{O}$ is the element of $SO(3)$, rotations of the three-dimensional real vector-space, corresponding to U in Eq. (1.73). This provides us with the opportunity of visualizing the action of the device as a rotation of a vector in the three-dimensional space. The detailed analysis can be found in Ref. [24]. The other important consequence is that the multiplets of the states are invariant subspaces of the beam splitter transformation in Eq. (1.73), namely, l is conserved by the transformation. Thus the notation in Eq. (1.78) is very suitable for the description of the beam splitter transformation.

1.3.9 Optical homodyning

Optical homodyning is a method for measuring quadratures $\hat{q}$, and $\hat{p}$ of Eq. (1.35) of a single mode light field. The idea is to employ a strong laser beam as phase reference to the light field to be measured. In a balanced homodyning scheme, the two fields interfere on a symmetric beam splitter, and intensities of the two output fields are measured. The difference of the two intensities can be easily shown to be proportional to one of the quadratures, depending on the phase shifts at the beam splitter. Homodyning is a fundamental tool of quantum state reconstruction. It is beyond the scope of this thesis to describe homodyning in detail. A good review can be found in Ref.[52].

Throughout this thesis, homodyne detectors will be considered as ideal devices realizing von Neumann measurement of a given quadrature.

1.3.10 Summary

In this chapter I have given a very brief introduction to the variety of tools applied throughout the rest of the thesis. The aim of this overview was to make this thesis as self-contained as possible, and provide the point of view most suitable for the phenomena discussed.

Chapter 2

Introduction to quantum teleportation

Quantum teleportation has been the subject of extremely intense research in the past few years. It would therefore be impossible to summarize even those considerations which are related to my results presented in this thesis. In this chapter I outline the very first ideas of quantum teleportation. Two schemes will be introduced here: the original Bennett scheme [7] for qubits, and the Braunstein-Kimble protocol for teleporting a state of a single-mode electromagnetic field [17, 34].

2.1 The Bennett scheme of teleportation

Let me describe the steps of a Bennett type teleportation scheme qualitatively. The sender – traditionally named Alice – possesses a physical system (subsystem 1, see Fig. 2.1) in some unknown state. The state has to be destroyed by Alice and "resurrected" at a receiver, Bob. Destroying the state is necessary, as copying ("cloning") of physical systems is forbidden in quantum mechanics [81]. In order to carry out teleportation, Alice and Bob share a bipartite system (subsystems 2 and 3), in an entangled state. Alice then carries out a joint measurement on subsystems 1 and 2, which entangles them. According to the von Neumann projection principle, subsystem 3 is left in some state which is a transformed version of the one to be teleported. The transformation itself depends on the outcome of Alice's measurement. This result is communicated to Bob, who obtains the teleported state by carrying out the inverse transformation.

Bennett formulated the protocol for qubits, and gave a generalization to arbitrary finite

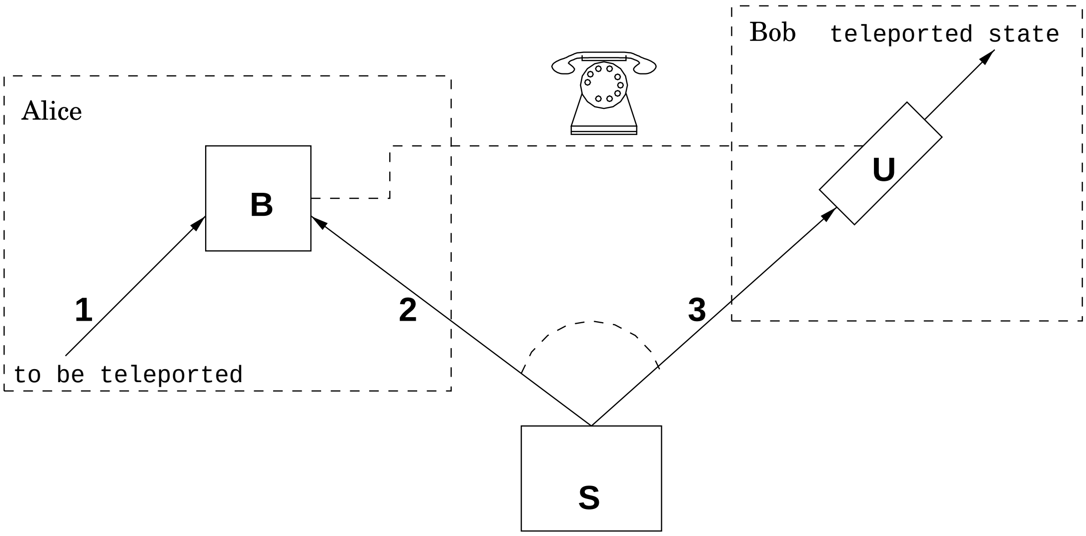

Figure 2.1: The block diagram of a general teleportation scheme. Alice and Bob share an entangled pair (entangled resource) emerging from the source S. Alice carries out a joint measurement entangling subsystems 1 and 2 with the device B, and communicates the result to Bob. After receiving this information Bob can restore the original state via a unitary transformation carried out by the device U. The required transformation depends on the classically communicated result.

dimensional schemes. I quote the description for pure states of qubits, but it is easy to see that it works for mixed states as well.

The Hilbert space of a single qubit is spanned by the computational basis $\{|0\rangle, |1\rangle\}$. Considering two qubits, the product basis of the joint Hilbert-space is $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, where we apply the notation $|ij\rangle = |i\rangle \otimes |j\rangle$ for tensor products. It is worth introducing another basis on the two-qubit Hilbert space, consisting of entangled states only. This is the so-called Bell-basis:

$$\begin{aligned} |\Psi^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle), \\ |\Phi^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle). \end{aligned} \tag{2.1}$$

These states are *maximally entangled*, which means that the partial trace of them in any of the subsystems is the completely mixed state represented by the density operator proportional to unity.

As for the teleportation protocol, Alice has subsystem 1 in the general state

$$|\Psi\rangle_1 = \alpha|0\rangle_1 + \beta|1\rangle_1, \qquad \beta = \sqrt{1-\alpha^2}e^{i\varphi}. \tag{2.2}$$

Alice and Bob share subsystems 2 and 3 in the entangled state

$$|\Psi^-\rangle_{23} = \frac{1}{\sqrt{2}}\left(|01\rangle_{23} - |10\rangle_{23}\right). \tag{2.3}$$

Now the state of the whole system of three qubits is

$$|\Psi\rangle_{123} = \frac{\alpha}{\sqrt{2}}(|001\rangle - |010\rangle) + \frac{\beta}{\sqrt{2}}(|101\rangle - |110\rangle). \tag{2.4}$$

Expanding this state on the Bell basis in subsystems 1 and 2, one obtains

$$\begin{aligned} |\Psi\rangle_{123} = \ & \tfrac{1}{2}(|\Psi^-\rangle_{12}\left(-\alpha|0\rangle_3 - \beta|1\rangle_3\right) + |\Psi^+\rangle_{12}\left(-\alpha|0\rangle_3 + \beta|1\rangle_3\right) + \\ & |\Phi^-\rangle_{12}\left(\alpha|1\rangle_3 + \beta|0\rangle_3\right) + |\Phi^+\rangle_{12}\left(\alpha|1\rangle_3 - \beta|0\rangle_3\right)). \end{aligned} \tag{2.5}$$

One can see that if Alice performs a joint measurement on subsystems 1 and 2 which projects onto the Bell basis, the state of system 3 becomes a unitary transform of the incoming state. The measurement result is transmitted to Bob through a classical channel. Bob, being aware of the result, can invert the appropriate unitary transformation, thereby succeeding in teleportation. For lack of the knowledge of the measurement result, it is impossible for Bob to restore the state. All four outcomes of the measurement appear with equal probability, thus if one observes the classical channel which transmits the measurement results, one finds two qubits of random noise. Both the classical information and quantum entanglement are *required* to teleport the state. As Bell states are maximally entangled, subsystem 1 remains in a maximally mixed state after the teleportation, thus the input state was indeed completely destroyed, and generated elsewhere.

Notice that if the measurement projected onto state $|\Psi^-\rangle$, Bob obtains the teleported state without doing any transformations. Thus in $1/4$ of the cases, teleportation succeeds even if Bob does not have an apparatus for carrying out unitary transformations. This is the case of *probabilistic* teleportation.

The scheme described here was first realized experimentally in Innsbruck [15], in 1997, and a similar one was carried out in Italy [12]. In both experiments, the state of a photon generated by down-conversion was teleported.

2.2 The Braunstein-Kimble scheme of teleportation

The scenario for teleporting continuous quantum variables was introduced theoretically, and realized experimentally first by Braunstein and Kimble [17, 34], based on the idea of Vaidman [73]. The protocol itself is very similar to the one depicted in Fig. 2.1: only the physical systems and sub-procedures involved are different.

The system Alice wants to teleport is a single-mode electromagnetic field in the state described by a Wigner function $W_1(q_1, p_1)$. The entangled resource is a two-mode electromagnetic field, in a so-called two-mode squeezed state

$$\begin{aligned} W_{23}(q_2, p_2, q_3, p_3) = \quad & \tfrac{4}{\pi^2} \exp(-e^{-2r}\left((q_2-q_3)^2+(p_2+p_3)^2\right) \\ & -e^{2r}\left((q_2+q_3)^2+(p_2-p_3)^2\right)), \end{aligned} \tag{2.6}$$

where r is called the *squeezing parameter*. In the ideal case r tends towards infinity, and

$$W_{23}(q_2, p_2, q_3, p_3) \rightarrow const. \cdot \delta(q_2+q_3)\delta(p_2-p_3). \tag{2.7}$$

This Wigner function clearly describes the original Einstein-Podolsky-Rosen situation [31], as it was discussed by Bell [6]: positions and momenta are completely correlated. However, the $r \to \infty$ situation cannot be achieved experimentally, as it would describe a state of infinite energy. In spite of that, I describe the simpler ideal case here.

The measuring device consists of a beam-splitter to entangle modes 1 and 2, and two homodyne detectors measuring quadratures of the modes. Two joint, commuting operators are measured, which are the results of a canonical transformation, e.g.,

$$Q = \frac{1}{\sqrt{2}}(q_2 + q_1), \qquad P = \frac{1}{\sqrt{2}}(p_2 - p_1). \tag{2.8}$$

This measurement is described in terms of the concepts of section 1.3.5: projection is described by averaging over the complementary variables of the measured quantities, partial traces are described by averaging over the corresponding subsystems. The measurement statistics of P and Q are found to be uniform: any value of Q and P can appear with equal probability. These values are communicated to Bob through the classical channel. After tracing (i.e. averaging) over mode 1, the (unnormalized) Wigner function of the output state is

$$W_3(q_3, p_3) = W_1(q_3 - Q, p_3 - P). \tag{2.9}$$

The output state is a shifted version of the input. In the non-ideal case, a Gaussian smoothing would appear. Bob can do the inverse shift (unitary transformation), and regain the required state.

Some features of the qubit scheme described in the previous subsection can be recognized: the classical information is a random noise, but teleportation cannot succeed without this information. Also, the input state is destroyed.

I gave a rather brief description of continuous variable teleportation here, a more detailed analysis will be a result of chapter 4.

2.3 Summary

I have introduced the two most typical protocols of quantum teleportation. The connection between the two protocols is not completely trivial. It will be better understood in section 3.4.

Chapter 3

Quantum teleportation on generic Hilbert-spaces

3.1 Introduction

This chapter describes some results concerning quantum teleportation on generic Hilbert spaces, valid regardless to the actual physical realization of the protocol.

In section 3.2, I investigate a classical limit of the Bennett scheme described in section 2.1. I show that quantum teleportation is a quantum generalization of the classic one-time pad cypher.

In section 3.3, after introducing some concepts of quantum channel theory, a rather abstract description of the most general finite dimensional Bennett-type quantum teleportation schemes is given. The description introduced is completely independent of the dimensions of the Hilbert spaces involved, and we do not even need to fix a basis. I give a general requirement of successful conditional teleportation in terms of the applicable entangled states and joint measurements.

In section 3.4 I give a description of Bennett's scheme in terms of the finite dimensional Wigner function formalism introduced by Wootters. The infinite dimensional limit is also outlined. These results reveal the connection between the Bennett scheme and the Braunstein-Kimble schemes outlined in chapter 2.

3.2 Classical limit of quantum teleportation

In this section I examine what happens to a teleportation scheme, when the density matrix elements of the entangled state used as an entangled resource, which are off-diagonal on the product state basis, are reduced. This approach, turning the entangled state into a classical correlation, obviously offers a possible way of obtaining a classical limit of the teleportation process.

In section 3.2.1 the class of bipartite states in argument is described, and Bennett's protocol of quantum teleportation of a qubit is summarized in a consistent density matrix formalism. The latter can be regarded as a special case of the treatment of [57] or [39]. By replacing the ideal EPR pair with the states investigated, I obtain the main result of the section. Starting from this, two examples are studied in detail: in Subsection 3.2.2 the purely classical limit is introduced, and in Subsection 3.2.3 cases between the ideal quantum teleportation and the classical limit are analyzed by examining a gedanken experiment.

Throughout this section, I use the terminology of spin-$\frac{1}{2}$ particles, which are prototypes of two-level quantum systems, that is, qubits. It will enable us to envisage the classical-quantum transition through a simple gedanken experiment. The spin z-component eigenstates of the particles are denoted by $|\uparrow\rangle$ and $|\downarrow\rangle$. This will be useful in order to describe the gedanken experiment in Subsection 3.2.3.

3.2.1 Quantum teleportation revisited

Let us examine Bennett's scheme of quantum teleportation [7] of a qubit: the sender, Alice has particle 1 of spin $\frac{1}{2}$ in the state

$$\rho_{\text{in}}^{(1)} = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}, \tag{3.1}$$

and wants to teleport it to Bob. The upper indices of density matrices (and other operators) refer to the number assigned to the particles. A and B share particles 2 and 3, as an entangled resource. These particles are in a state $\rho^{(23)}$, which is a pure EPR singlet in the ideal protocol. There is also a classical communication channel between the parties. Alice has an ideal Bell-state detector, and Bob can carry out unitary transformations on particle 3 given to him. Let us suppose that the state of particles 2 and 3 is described by the following density matrix:

$$\rho_{\text{shared}}^{(23)}(\alpha) = \frac{1}{2}(\left|\uparrow_2\right\rangle\left|\downarrow_3\right\rangle\left\langle\uparrow_2\right|\left\langle\downarrow_3\right| + \left|\downarrow_2\right\rangle\left|\uparrow_3\right\rangle\left\langle\downarrow_2\right|\left\langle\uparrow_3\right|$$
$$-\alpha\left|\uparrow_2\right\rangle\left|\downarrow_3\right\rangle\left\langle\downarrow\right|_2\left\langle\uparrow_3\right| - \alpha\left|\downarrow_2\right\rangle\left|\uparrow_3\right\rangle\left\langle\uparrow_2\right|\left\langle\downarrow_3\right|). \tag{3.2}$$

The lower indices in the kets indicate the number assigned to the particles. Let α be a real parameter between 0 and 1. For $\alpha = 1$, $\rho_{\text{shared}}^{(23)}(1) = \left|\Psi_{23}^{(-)}\right\rangle\left\langle\Psi_{23}^{(-)}\right|$, thus in this case Alice and Bob share an ideal EPR pair. This is the case of ideal quantum teleportation. Otherwise, the off-diagonal elements of the density matrix are multiplied by α. For $\alpha = 0$ the density matrix is diagonal in the product state basis, describing classical statistics only. This state is the mixture of the product states $\left|\uparrow_2\right\rangle\left|\downarrow_3\right\rangle$ and $\left|\downarrow_2\right\rangle\left|\uparrow_3\right\rangle$. Such a state can be generated by a *classically stochastic* source emitting particles with opposite spins, with equal probability of sending “up” and “down” state both for particles 2 and 3.

The Bell-states introduced in Eq. 2.1 will be denoted by

$$\left|\Psi^{(\pm)}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle\left|\downarrow\right\rangle \pm \left|\downarrow\right\rangle\left|\uparrow\right\rangle\right)$$
$$\left|\Phi^{(\pm)}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle\left|\uparrow\right\rangle \pm \left|\downarrow\right\rangle\left|\downarrow\right\rangle\right) \tag{3.3}$$

here. The states in Eq. (3.2) may be rewritten in the Bell-basis, yielding

$$\rho_{\text{shared}}^{(23)}(\alpha) = \frac{1+\alpha}{2}\left|\Psi_{23}^{(-)}\right\rangle\left\langle\Psi_{23}^{(-)}\right| + \frac{1-\alpha}{2}\left|\Psi_{23}^{(+)}\right\rangle\left\langle\Psi_{23}^{(+)}\right|, \tag{3.4}$$

that is, the state is a mixture of the two Ψ Bell-states. These states are similar to Werner states (which were examined from the teleportation's point of view in references [35, 13]).

Let me consider now the entire teleportation process. Initially, the state of the whole system of three particles is the product of the states in Eqs. (3.1) and (3.2):

$$\rho_{\text{in}}^{(123)} = \rho_{\text{in}}^{(1)} \otimes \rho_{\text{shared}}^{(23)}(\alpha). \tag{3.5}$$

Alice carries out a Bell-state measurement, which is described by one of the operators including projection of subsystems 1 and 2 to a Bell-state,

$$\hat{P}_i^{(123)} = 4\left(\left|\Psi_{12}^{(i)}\right\rangle\left\langle\Psi_{12}^{(i)}\right| \otimes \hat{1}^{(3)}\right), \tag{3.6}$$

where $|\Psi^{(i)}\rangle$ stands for one of the four Bell-states. The result of the measurement is i, corresponding to the i-th Bell-state. This information is sent to Bob via the classical channel.

Because of the detection of each Bell-state occurs with probability $\frac{1}{4}$, the operator is multiplied by 4 in order to preserve the norm of the state obtained. The state of the system after the measurement is given by applying the operator in equation (3.6) to the state in equation (3.5). From this one obtains the state of Bob's particle by tracing out in the other two particles:

$$\rho_{\mathrm{u}}^{(3)} = \mathrm{Tr}_{12}\left(P_i^{(123)}\rho_{\mathrm{in}}^{(123)}P_i^{(123)}\right). \tag{3.7}$$

In the last step, Bob has to apply a unitary transformation $U^{(3)}$ on state $\rho_u^{(3)}$, according to Alice's measurement result i, which is the identity operator in case Alice has detected $|\Psi^{(-)}\rangle$, and

$$U^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{3.8}$$

in case of detecting $\left|\Psi^{(+)}\right\rangle, \left|\Phi^{(-)}\right\rangle$, and $\left|\Phi^{(+)}\right\rangle$ respectively. Carrying out the calculations described, one obtains the following result: the state teleported to Bob reads

$$\rho_{\mathrm{out}}^{(3)} = \begin{pmatrix} \rho_{00} & \alpha\rho_{01} \\ \alpha\rho_{10} & \rho_{11} \end{pmatrix}. \tag{3.9}$$

It can be seen that *the reduction of the off-diagonal elements of the density matrix describing the entangled resource is inherited by the teleported state.* Thus teleportation acts as a phase-damping channel [64].

3.2.2 The one-time-pad as a classical limit of teleportation

As a first example, let me examine the case of "teleporting" a classical bit. Assume that $\alpha = 0$, that is, the density matrix describing the entangled resource is diagonal. According to Eq. (3.9), only the diagonal matrix elements of the density matrix of the input state, i.e. the statistics of the measurement of the spin-z component are preserved. Let us suppose that the input state in Eq. (3.1) is already diagonal: $\rho_{10} = \rho_{01} = 0$, and consider measurement of the z components of the spins.

Under these circumstances our particles can be exactly identified with classical bits. The states of these classical bits, denoted by $\uparrow$, and $\downarrow$, are identical with the basis quantum states $|\uparrow\rangle$ and $|\downarrow\rangle$. The diagonal density matrix of the input quantum state describes a classical probability distribution of bit 1. This is transferred into bit 3 via a classical communication channel and a *classical* correlation. The process itself can be interpreted in the following way: The source of bits 2 and 3 generates correlated bit-pairs $\downarrow_2\uparrow_3$ or $\uparrow_2\downarrow_3$ with equal probability. This can be considered as the classical limit of an EPR-pair. Bit 2 is obtained by Alice, who makes a measurement, which tells whether bit 1 and 2 are the same. One cannot speak of superpositions in this classical context, and therefore the two Ψ and the two Φ Bell-states coincide in this limit: the former two mean simply "the two bits are different" (Ψ-detection), and the latter "the two bits are the same" (Φ-detection). Thus the Bell-state measurement degenerates to an "exclusive or" operation, resulting in a single bit of information, communicated to Bob. Bob has to carry out the proper transformation to regain the "teleported" bit. There is no phase of the probability amplitudes for classical probability, thus the transformations in Eq. (3.8) degenerate to a conditional NOT operation: in case of "Ψ detection", Bob has obtained bit 3 in the proper state, while in case of "Φ detection" he has to invert bit 3. Finally bit 3 is left with the original value of bit 1. Since the values of bits 1 or 2 are irrelevant for the "Bell-state measurement", it is not necessary for Alice to be aware of the actual value of bit 1 to be "teleported". Therefore the method works for "teleporting" an unknown classical bit as well, similarly to the quantum protocol.

The classical protocol described here is well known as the one-time-pad cypher in classical cryptography [67]. This is, in some sense, the classical protocol most similar to quantum teleportation. It is the "teleportation" of the (possibly unknown) state of a classical bit via a classical communication channel and a classical correlation. The classical correlation provides random noise without the classical communication channel, while the classical channel by itself is useless for reconstruction of the result without the member of the correlated pair. Note that instead of measuring the state of bit 1 and simply communicating its value, only a comparison with a reference has been made.

Figures 3.1 and 3.2 visualize the relation between the one-time-pad cypher and Bennett's quantum teleportation scheme.

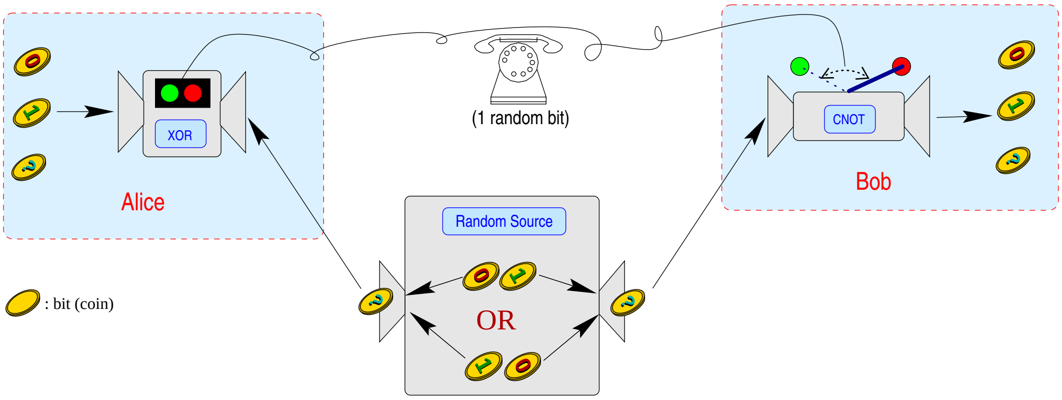

Figure 3.1: The one-time pad cypher

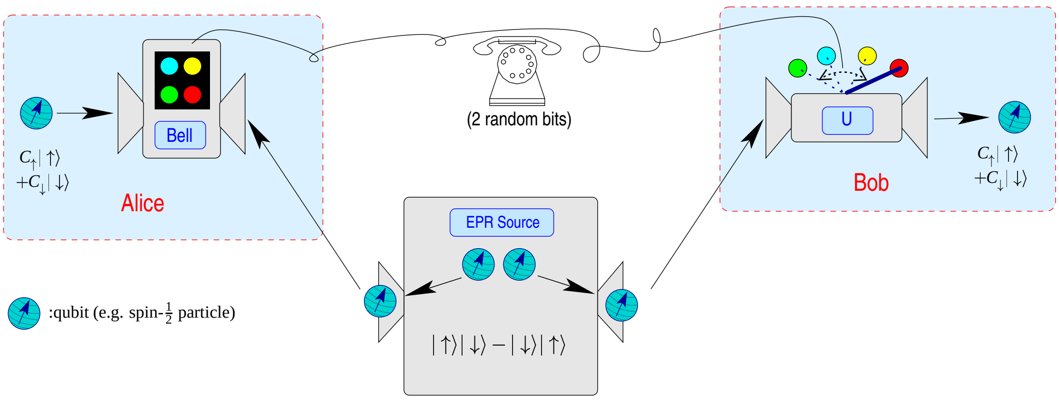

Figure 3.2: Bennett's scheme, to be compared with 3.1

3.2.3 Statistics of a gedanken experiment

Having described the classical analogue, I now examine the case $\alpha \neq 0$, which interpolates between the classical and the quantum case. I calculate the consequence of Eq. (3.9) to the result of a teleportation experiment. For simplicity, let me suppose that the spin $\frac{1}{2}$ state to be teleported is $|\uparrow\rangle$, rotated by a given angle φ around the x axis of the coordinate system. The operator of this rotation ($\hbar = 1$) is

$$\hat{R}(\varphi) = e^{\frac{i}{2}\varphi\sigma_x}, \tag{3.10}$$

σ_x being the first Pauli-matrix, and thus we have the state

$$|\Psi_{\text{in}}(\varphi)\rangle = \hat{R}(\varphi)\,|\uparrow\rangle = \begin{pmatrix} \cos(\frac{\varphi}{2}) \\ i\sin(\frac{\varphi}{2}) \end{pmatrix} \tag{3.11}$$

to be teleported. According to (3.9), one obtains

$$\rho_{\text{out}}(\varphi,\alpha) = \begin{pmatrix} \cos^2(\frac{\varphi}{2}) & -i\alpha\cos(\frac{\varphi}{2})\sin(\frac{\varphi}{2}) \\ i\alpha\cos(\frac{\varphi}{2})\sin(\frac{\varphi}{2}) & \sin^2(\frac{\varphi}{2}) \end{pmatrix} \tag{3.12}$$

as a result of the teleportation process. In order to verify the teleportation, one may measure the spin along an axis obtained by rotating the z axis by the angle φ_m around the x axis. The probability of finding a spin component of $+\frac{1}{2}$ along this direction is

$$\begin{aligned} \mathscr{P}(\varphi,\varphi_m,\alpha) &= \text{Tr}\left(\rho_{\text{out}}(\varphi,\alpha)\hat{R}(\varphi_m)\,|\uparrow\rangle\,\langle\uparrow|\,\hat{R}^\dagger(\varphi_m)\right) \\ &= \frac{\cos(\varphi)\cos(\varphi_m)}{2} + \frac{\alpha\sin(\varphi)\sin(\varphi_m)}{2} + 1/2. \end{aligned} \tag{3.13}$$

For $\alpha = 1$ the familiar cosine-type result valid for ideal teleportation is obtained,

$$\mathscr{P}(\varphi,\varphi_m,1) = \frac{\cos(\varphi-\varphi_m)+1}{2}, \tag{3.14}$$

which is equal to 1 for $\varphi = \varphi_m$, meaning perfect teleportation to any direction. For a given input state $|\Psi_{\text{in}}(\varphi)\rangle$, the probability for finding the output in the input state after the teleportation is the fidelity of the teleportation of the input state. This fidelity is

$$\mathscr{P}(\varphi,\varphi,\alpha) = \frac{\alpha-1}{2}\sin^2\varphi + 1. \tag{3.15}$$

In figure 3.3 I have plotted this function. It is equal to 1 in the case of ideal teleportation. The basis states $|\uparrow\rangle$ and $|\downarrow\rangle$ are always properly teleported, as we have seen in the classical

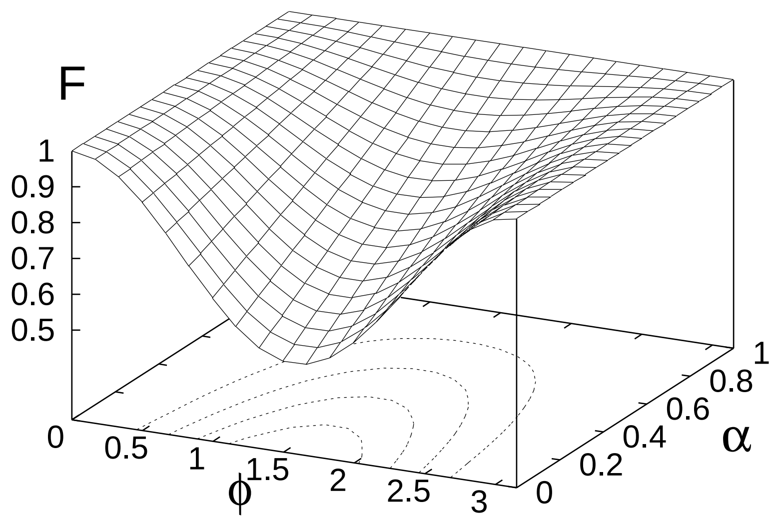

Figure 3.3: The fidelity of the teleportation of the spin pointing to the direction described by the angle φ plotted against the parameter α describing the impurity of the state.

case. The minimum of the fidelity for any α occurs for the equal superposition of the two basis states ($\varphi_m = \pi/2$). The minimum value for $\alpha = 0$ is $\frac{1}{2}$, expressing that the result of the measurement can be either $\uparrow$ or $\downarrow$ with equal probability, thus this state is not teleported at all. Increasing the purity of the entangled resource the domain of the angles increases, in which teleportation can be regarded as reliable.

The consideration presented here illustrates the quantum-to-classical transition.

3.3 Teleportation in terms of relative state representations

In this section finite dimensional probabilistic teleportation schemes will be classified generally. I utilize elements of the theory of quantum channels. “Quantum channel” is the synonym for “quantum operation”, though this word is sometimes used in a different sense in the literature. The term “superoperator” is also used in the same sense.

3.3.1 States, channels and antilinear maps

In classical communication schemes a *channel* carries information from one place to another. Thus a channel is modeled by a function describing the changes the information is subjected to by the transmission. The quantum counterpart is described as follows. Let A be a physical system with the Hilbert space $\mathscr{H}_A$. Let the set of all states of the system, that is, the set of possible density operators be denoted by $\mathscr{S}_A$. We suppose that a state $\rho \in \mathscr{S}_A$ is transmitted to a state $\rho' \in \mathscr{S}_A$ on the same Hilbert space: the transmitted state should be a substitute of the initial in the optimal case. (We may consider a target space $\mathscr{H}_B$ isomorph to $\mathscr{H}_A$ to be more accurate.) Generally, ρ' may be any state in $\mathscr{S}_A$, a quantum channel is the most general evolution of a quantum state: it transforms density matrices into other density matrices.

It can be shown that a quantum channel is mathematically a completely positive, trace-preserving, and hermiticity preserving $\mathscr{S}_A \to \mathscr{S}_A$ linear map, which we will denote by $\$_A$. These properties are required to map a density matrix to another. The quantum channel acts as $\$_A(\rho) = \rho'$. Note that quantum teleportation protocols are also quantum channels, taking the state to be teleported into the teleported state.

In all the possible cases quantum channels can also be regarded as if they described the evolution of a subsystem of a large system subjected to unitary evolution.

There are several ways of representing a quantum channel. One of them, the so-called relative state representation [68] is an excellent tool for our purposes.

Consider an ancillary system B in addition to A, with Hilbert space $\mathscr{H}_B$ with the same finite dimensionality N as $\mathscr{H}_A$. Let $\{|i\rangle_A\}_{(i=0,\ldots,N-1)}$ and $\{|i\rangle_B\}_{(i=0,\ldots,N-1)}$ denote orthonormal bases on $\mathscr{H}_A$ and $\mathscr{H}_B$.

$$\left|\Psi^+\right\rangle_{AB} = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle_A \otimes |i\rangle_B \tag{3.16}$$

is a pure, maximally entangled state of the system. All other maximally entangled states can be obtained from $|\Psi^+\rangle$ by local unitary transformations. (A local unitary transformation acts on one of the subsystems only.)

Any pure state $|\Psi\rangle_A \in \mathscr{H}_A$ can be described by an (unnormalized) *index state* $|\Psi^*\rangle_B \in \mathscr{H}_B$ such that

$$|\Psi\rangle_A = N\, {}_B\langle\Psi^*|\Psi^+\rangle_{AB}. \tag{3.17}$$

The state in argument is obtained as a partial inner product of its index state and the maximally entangled state $|\Psi^+\rangle$. The normalization factor N is present for notational convenience, its role will become clear later. Representing a state with the corresponding index state is called the *relative state representation of the state*. It depends on the entangled state $|\Psi^+\rangle_{AB}$.

The mapping

$$L_{|\Psi^+\rangle} : \mathscr{H}_A \to \mathscr{H}_B, \quad L_{|\Psi^+\rangle}|\Psi\rangle_A = |\Psi^*\rangle_B, \tag{3.18}$$

creating the index state from the original state will have a crucial role in our considerations. It is antilinear (conjugate linear), which means

$$L_{|\Psi^+\rangle}\left(\sum_i C_i\,|\psi_i\rangle\right) = \sum_i C_i^* L_{|\psi^+\rangle}\,|\psi_i\rangle \tag{3.19}$$

for any set of C_i-s and $|\psi_i\rangle$-s. In fact $\sqrt{N}L_{|\Psi^+\rangle}$ is also antiunitary: for any $|\psi\rangle$ and $|\varphi\rangle$, with $|\psi'\rangle = \sqrt{N}L_{|\Psi^+\rangle}\,|\psi\rangle$, and $|\varphi'\rangle = \sqrt{N}L_{|\Psi^+\rangle}\,|\varphi\rangle$,

$$\langle\varphi'|\psi'\rangle = (\langle\varphi|\psi\rangle)^* \tag{3.20}$$

holds. Indeed, expanding an arbitrary $|\Psi\rangle_A$ on the computational basis,

$$L_{|\Psi^+\rangle}|\Psi\rangle_A = L_{|\Psi^+\rangle}\sum_i C_i|i\rangle_A = \frac{1}{\sqrt{N}}\sum_i C_i^*|i\rangle_B, \tag{3.21}$$

from which the above properties follow.

Let us now see how to utilize the relative state representation of states to represent quantum channels. Let us have the compound system in the state $|\Psi^+\rangle_{AB}$, and send subsystem A through the channel $\$_A$ while doing nothing with subsystem B. The effect of the channel on any pure state $|\Psi\rangle_A$ of system A is then obtained by the partial inner product with the corresponding index state:

$$\$_A\left(|\Psi\rangle_{A\,A}\langle\Psi|\right) = N^2\,{}_B\langle\Psi^*|(\$_A\otimes\hat{1}_{\mathscr{S}_B})\left(|\Psi^+\rangle_{AB\,AB}\langle\Psi^+|\right)|\Psi^*\rangle_B, \tag{3.22}$$

where $|\Psi^*\rangle_B = L_{|\Psi^+\rangle}|\Psi\rangle_A$. Thus we have represented the channel $\$_A$ with the bipartite state

$$\rho_{AB} = (\$_A\otimes\hat{1}_{\mathscr{S}_B})\left(|\Psi^+\rangle_{AB\,AB}\langle\Psi^+|\right). \tag{3.23}$$

The effect of the channel on any state can be obtained from this fixed state. The relative state representation of the channel of the bipartite state has a completely mixed partial trace:

$$\mathrm{tr}_A \rho_{AB} = \frac{1}{N}\hat{1}_B. \tag{3.24}$$

This follows from the fact that $|\Psi^+\rangle_{AB}$ has a maximally mixed partial trace, and this property cannot be changed by local (product-form) operations such as $\$_A \otimes \hat{1}_B$, and thus (3.24) will also hold for the ρ_{AB} obtained in Eq. (3.23).

Thus far we have shown that a channel can be represented by a bipartite density matrix of completely mixed partial trace. The converse statement is also true: for all bipartite states with completely mixed partial trace, there exists a channel. In fact, an isomorphism between the set of all $\$_A$ channels on $\mathscr{S}_A$, and the set of bipartite states $\rho_{AB} \in \mathscr{S}_{\mathscr{H}_A \otimes \mathscr{H}_B}$ with maximally mixed partial traces can be found similarly to Eq. (3.22). The bipartite state corresponding to a channel can be obtained from $|\Psi^+\rangle$ by applying the channel on system A and doing nothing with system B. This isomorphism has been found by Horodecki *et al.* [39], who have discussed it in detail, and have used it for the description of teleportation processes as quantum channels.

Notice that the relative state representation relies on the entangled state $|\Psi^+\rangle$, or otherwise, the map $L_{|\Psi^+\rangle}$ of Eq. (3.18). Having fixed these, states and channels can be described conveniently. Let us follow a more general way now. We may use the set of antilinear $\mathscr{H}_A \rightarrow \mathscr{H}_B$ maps in order to describe pure states in $\mathscr{H}_A \otimes \mathscr{H}_B$. As relative state representation is also based on L, changing this map can give rise to different relative state representations.

Consider a bipartite pure state $|\Phi\rangle_{AB} \in \mathscr{H}_A \otimes \mathscr{H}_B$. We may expand it on the computational basis as

$$|\Phi\rangle_{AB} = \sum_{ij} C_{ij} |i\rangle_A \otimes |j\rangle_B. \tag{3.25}$$

We define the $\mathscr{H}_A \rightarrow \mathscr{H}_B$ antilinear operator $L_{|\Phi\rangle}$ such that

$$L_{|\Phi\rangle} |i\rangle_A = \sum_j C_{ij} |j\rangle_B. \tag{3.26}$$

Thus we can write

$$|\Phi\rangle_{AB} = \sum_i |i\rangle_A \otimes (L_{|\Phi\rangle} |i\rangle_A). \tag{3.27}$$

For any bipartite pure state $|\Phi\rangle_{AB} \in \mathscr{H}_A \otimes \mathscr{H}_B$, there exists a unique antilinear operator $L_{|\Phi\rangle}$ defined this way.

The representation of a state $|\Phi\rangle_{AB}$ with $L_{|\Phi\rangle}$ has the advantage that both $L_{|\Phi\rangle}$, and the expression in Eq. (3.27) relating the antiunitary operator to the corresponding state are independent of the actual computational basis chosen on $\mathscr{H}_A$. This can be shown as follows: suppose that we chose another basis on $\mathscr{H}_A$:

$$\left|\tilde{i}\right\rangle_A := \sum_i U_{\tilde{i},i} \left|i\right\rangle_A , \tag{3.28}$$

where U is an arbitrary unitary matrix. We can substitute the basis $|\tilde{i}\rangle_A$ into Eq. (3.27):

$$\begin{aligned} \sum_{\tilde{i}} |\tilde{i}\rangle_A \otimes (L_{|\Phi\rangle} |\tilde{i}\rangle_A) = \sum_{\tilde{i}} \left(\sum_i U_{\tilde{i},i} |i\rangle_A \right) \otimes \left(L_{|\Phi\rangle} \sum_j U_{\tilde{i},j} |j\rangle_A \right) = \\ \sum_{\tilde{i}} \sum_{i,j} U_{\tilde{i},i} U^*_{\tilde{i},j} \Big(|i\rangle_A \otimes (L_{|\Phi\rangle} |j\rangle_A) \Big) = \sum_{i,j} \left(U^\dagger U \right)_{i,j} \Big(|i\rangle_A \otimes (L_{|\Phi\rangle} |j\rangle_A) \Big) = \\ \sum_{\tilde{i}} |i\rangle_A \otimes (L_{|\Phi\rangle} |i\rangle_A), \end{aligned} \tag{3.29}$$

where we have exploited the antilinearity of $L_{|\Phi\rangle}$. This invariance property makes the representation of bipartite pure states by antilinear operators extremely handy: possessing the operator $L_{|\Phi\rangle}$ one may calculate on any basis without the need of any transformation of the operator itself.

The representation introduced here possesses utmost pleasant properties. In order to see this, consider the set of bounded antilinear operators $L\colon \mathscr{H}_A \to \mathscr{H}_B$. We first define the adjoint $L^\star$ of an antilinear operator L so that for any $|\psi\rangle_A \in \mathscr{H}_A$ and $|\varphi\rangle_B \in \mathscr{H}_B$,

$$\left(L\,|\psi\rangle_A\right)^\dagger |\varphi\rangle_B = \left({}_A\langle\psi| L^\star |\varphi\rangle_B\right)^* . \tag{3.30}$$

Then one may define a scalar product of two operators

$$(L', L) = \mathrm{Tr}(L^\star L') \tag{3.31}$$

which is antilinear in the first argument. This generates the norm

$$\|L\|_\star = \sqrt{(L,L)} \tag{3.32}$$

Let $\mathscr{C}_{AB}$ denote the set of bounded antilinear operators $L\colon \mathscr{H}_A \to \mathscr{H}_B$ which have finite norm

$$\mathscr{C}_{AB} = \{L\colon \mathscr{H}_A \to \mathscr{H}_B \text{ antilinear} \,|\, \|L\|_\star < \infty\}. \tag{3.33}$$

With the above norm and scalar product, $\mathscr{C}_{AB}$ forms a Hilbert space. It is shown in Ref. [2] that (3.27) establishes a unitary isomorphism between $\mathscr{C}_{AB}$ and $\mathscr{H}_A \otimes \mathscr{H}_B$ in a natural way. Every pure bipartite state $|\Phi\rangle_{AB}$ can uniquely be described by an antilinear operator $L_{|\Phi\rangle} \in \mathscr{C}_{AB}$ such that $\mathrm{tr}(L^\star_{|\Phi\rangle} L_{|\Phi\rangle}) = 1$. Conversely, every such L describes a pure bipartite state.

Let us further investigate the properties of the adjoint operator defined in Eq. (3.30). Taking the representation of $L_{|\Psi\rangle_{AB}}$ on an orthonormal basis in Eq. (3.26), it can be shown, that $L^\star$ is the antilinear operator

$$L^\star_{|\Phi\rangle_{AB}} |j\rangle_B = \sum_i C_{ij} |i\rangle_A . \tag{3.34}$$

The "matrix" $C_{i,j}$ of L becomes transposed for the adjoint operator. With the aid of (3.34) it is straightforward to show that the $L^\star_{|\Phi\rangle_{AB}} L_{|\Phi\rangle_{AB}}$ and $L_{|\Phi\rangle_{AB}} L^\star_{|\Phi\rangle_{AB}}$ $\mathscr{H}_A \to \mathscr{H}_A$ and $\mathscr{H}_B \to \mathscr{H}_B$ operators are both linear and Hermitian. In fact, comparing with Eq. (3.25) one obtains that

$$\begin{aligned} L^\star_{|\Phi\rangle_{AB}} L_{|\Phi\rangle_{AB}} &= \mathrm{Tr}_B |\Phi\rangle_{AB\,AB}\langle\Phi| = \rho_A \\ L_{|\Phi\rangle_{AB}} L^\star_{|\Phi\rangle_{AB}} &= \mathrm{Tr}_A |\Phi\rangle_{AB\,AB}\langle\Phi| = \rho_B, \end{aligned} \tag{3.35}$$

that is, these operators are the density matrices of the subsystems. It is therefore very easy to calculate partial traces in our representation.

According to Eq.3.35, it is now straightforward to characterize *maximally entangled* pure states and this will characterize all the possible relative state representations as well. As maximally entangled states are of maximally mixed partial trace (3.24), a bipartite state is maximally entangled if and only if for the corresponding antilinear operator $LL^\star = N^{-1}I_B$ and $L^\star L = N^{-1}I_A$. This is equivalent to the statement that $\sqrt{N}L$ is *antiunitary*. On the other hand, the operator $L_{|\Phi\rangle}$ gives rise to a relative state representation if and only if $\{L_{|\Phi\rangle}|i\rangle\}_{i=0,\dots,N-1}$ forms an orthogonal basis on $\mathscr{H}_B$ where in addition $\sqrt{N}L_{|\Phi\rangle}$ is antiunitary. We can conclude that a pure state is maximally entangled if and only if the corresponding antilinear operator $\sqrt{N}L$ is antiunitary, and relative state representations can be defined via maximally entangled states only.

3.3.2 General probabilistic teleportation

Let me apply the antilinear description of bipartite states introduced in the last section for quantum teleportation. Suppose that system A at Alice, prepared in the unknown state $|\Phi\rangle_A$ is to be teleported, and systems B and C shared by the parties (Alice and Bob) are in a (possibly only partially entangled) state $|\sigma\rangle_{BC}$. I will call this the *shared* state in what follows. The shared state is described by the antilinear map $L_{|\sigma\rangle}$. Systems A and B are located at Alice who performs a joint projective measurement on them. Suppose that its outcome corresponds to the projection onto the state $|\sigma_q\rangle_{AB}$. In the followings, I will regard only this outcome, thus our teleportation scheme will be a probabilistic, conditional one.

To have common computational bases in the description of the shared state, and the state Alice's measurement projects onto, one expands $|\sigma_q\rangle$ in the following way:

$$|\sigma_q\rangle_{AB} = \sum_i (L_q|i\rangle) \otimes |i\rangle_B, \tag{3.36}$$

where $L_q \in \mathscr{C}_{BA}$. This is fundamentally the same as the representation of Eq. (3.27): the only difference is that the subsystems are labelled in the reverse order. One can represent the state corresponding to nondegenerate measurement outcome q, i.e., the bipartite state the measurement projects onto if the result is q, by a bounded antilinear operator $L_q\colon \mathscr{H}_B \to \mathscr{H}_A$. Note that L_q is unique disregarding a unit complex phase factor.

On the Hilbert space $\mathscr{C}_{BA}$ (3.33), the set of projectors corresponding to a measurement of any nondegenerate joint observable of A and B, is (up to phase factors) uniquely described by an orthonormal basis in $\mathscr{C}_{BA}$. Those measurements whose nondegenerate outcomes are represented by projections onto mutually orthogonal maximally entangled states, are called measurements of Bell type [30]. Every Bell measurement can be described by an orthonormal basis $\{L_q\}_{q=0\ldots\dim\mathscr{H}_A-1}$ in $\mathscr{C}_{BA}$, so that $\sqrt{\dim\mathscr{H}_a}L_q$ is antiunitary for every q.

Now I will calculate the teleportation channel, or rather the function $f_q\colon \mathscr{H}_A \rightarrowtail \mathscr{H}_C$ that relates the input state and the state of system C after the Bell measurement of Alice. Note that, although the unitary transformation that Bob carries out to obtain the exact teleported state is usually also included in the definition of teleportation channel, this terminology is more convenient here, as I investigate linearity and reversibility. At the beginning, the three systems are in the state $|\Phi\rangle_A \otimes |\sigma\rangle_{BC}$. The probability of the outcome q under consideration is given by

$$\begin{aligned} p_q(|\Phi\rangle_A) &= \left\| [(|\sigma_q\rangle_{AB\,AB}\langle\sigma_q|) \otimes I_C)](|\Phi\rangle_A \otimes |\sigma\rangle_{BC}) \right\|^2 = \left\| \sum_i ({}_A\langle\Phi|L_q|i\rangle_B^*) L|i\rangle_B \right\|^2 \\ &= \left\| \sum_i L(|i\rangle_{B\,B}\langle i|L_q^\star|\Phi\rangle_A) \right\|^2 = \left\| LL_q^\star|\Phi\rangle_A \right\|^2. \end{aligned} \tag{3.37}$$

On condition that the measurement yields the outcome q, the state of system C can be written as

$$\frac{1}{\sqrt{p_q(|\Phi\rangle_A)}} \sum_i ({}_{AB}\langle\sigma_q|\Phi\rangle_A|i\rangle_B) L|e_i\rangle_B = \frac{1}{\sqrt{p_q(|\Phi\rangle_A)}} LL_q^\star|\Phi\rangle_A. \tag{3.38}$$

The teleportation channel for the outcome q is

$$f_q\colon \mathscr{H}_A \rightarrowtail \mathscr{H}_C, \quad f_q(|\Phi\rangle_A) = \frac{LL_q^\star|\Phi\rangle_A}{\left\|LL_q^\star|\Phi\rangle_A\right\|}. \tag{3.39}$$

If the input state is given by the density operator ρ_{in} then the probability of the outcome q is

$$p_q(\rho_{\text{in}}) = \text{tr}_A\left(L_q L^\star L L_q^\star \rho_{\text{in}}\right) \tag{3.40}$$

and the output state is

$$\rho_{\text{out}} = \frac{LL_q^\star \rho_{\text{in}} L_q L^\star}{\text{tr}_A(L_q L^\star L L_q^\star \rho_{\text{in}})}. \tag{3.41}$$

I have defined a special quantum operation based on the teleportation scheme of Ref. [7]. One can obtain from (3.40) that this operation is a generalized (POVM) measurement of the input state. The POVM outcome is $L_q L^\star L L_q^\star$.

The channel f_q has to be reversible, so that I can obtain a teleported state identical to the original input state. I call the channel f_q reversible, if it is injective, that is, for different input states $|\Phi\rangle_A$ ($\||\Phi\rangle_A\| = 1$) the corresponding output states $f_q(|\Phi\rangle_A)$ are different. I remark that the reversibility of teleportation channels has also been investigated in Ref. [57]. I adopt a more general definition here. Reversibility means that every input state can be recovered (theoretically) from the output state. One can easily verify that this condition is equivalent to the requirement that the linear operator $LL_q^\star\colon \mathscr{H}_A \to \mathscr{H}_C$ should injective.

It may be the case, however, that the channel f_q is not linear. In this way, the input state can be recovered from the output only by using some nonlinear transformations, which may be unrealistic in some cases. Therefore, it is a natural requirement for the channel to be linear.

I show that if the teleportation channel is reversible, then its linearity is equivalent to the property that the probability (3.37) of the outcome q is independent of the input state $|\Phi\rangle_A$. Suppose that $|\Phi\rangle_1$ and $|\Phi\rangle_2$ are linearly independent, and let $(\alpha_1|\Phi\rangle_1+\alpha_2|\Phi\rangle_2)$ be of unit norm. From the linearity condition $f_q(\alpha_1|\Phi\rangle_1+\alpha_2|\Phi\rangle_2) = \alpha_1 f_q(|\Phi\rangle_1) + \alpha_2 f_q(|\Phi\rangle_2)$, one can obtain:

$$\alpha_1\left(\frac{1}{\|LL_q^\star(\alpha_1|\Phi\rangle_1+\alpha_2|\Phi\rangle_2)\|}-\frac{1}{\|LL_q^\star|\Phi\rangle_1\|}\right)LL_q^\star|\Phi\rangle_1$$
$$+\alpha_2\left(\frac{1}{\|LL_q^\star(\alpha_1|\Phi\rangle_1+\alpha_2|\Phi\rangle_2)\|}-\frac{1}{\|LL_q^\star|\Phi\rangle_2\|}\right)LL_q^\star|\Phi\rangle_2=0. \tag{3.42}$$

Since f_q is injective, $LL_q^\star|\Phi\rangle_1$ and $LL_q^\star|\Phi\rangle_2$ are also linearly independent. Then (3.42) implies that their coefficients are zero, that is, the probability (3.37) of the outcome q is independent of the input state $|\Phi\rangle_A$. Conversely, if (3.37) is independent of $|\Phi\rangle_A$, then $LL_q^\star$ is injective and f_q is linear. One can conclude that the condition "the probability of the measurement outcome q does not depend on the input state" (that is, Alice learns nothing about the input state due to the measurement) is equivalent to the linearity of the teleportation channel. Moreover, it can be proven in a way not detailed here that the linearity of the channel is equivalent to its unitarity—therefore its unitary reversibility.

Summarizing the results presented in this section, I gave a very compact description of a quantum teleportation process in Eqs. (3.39),(3.40), and3.41). This gives a description of a conditional teleportation scheme utilizing an arbitrary pure entangled resource. Both the entangled resource and the state Alice's measurement projects onto is described in the convenient antilinear operator formalism, which is completely basis-independent. These results were the basis of a consideration, which gave the condition for probabilistic teleportation with nonmaximally entangled states [50].

3.4 Quantum teleportation in terms of discrete Wigner functions

All quantum mechanical phenomena may be described in terms of quasiprobability distributions, as an alternative to the direct application of density matrices. Wigner functions are especially frequently applied, as they behave similarly to classical probability distributions from several points of view. For quantum states with infinite dimensional Hilbert spaces, the application of Wigner functions, as described in section 1.3.5, has become a standard part of considerations. For finite dimensional Hilbert spaces, the Wigner function formalism was first proposed by Wootters [80]. The discrete Wigner functions have proven to be useful in investigating coherent states in a finite-dimensional basis [19], definition of Q-functions and other propensities [58], and also played a role in the development of number-phase Wigner functions [72]. Quantum tomography for finite-dimensional Wigner functions has also been developed, applying a generalized definition [51]. After the appearance of my paper containing the following results, the application of discrete Wigner functions in quantum information developed further [8, 61]

A conspicuous question is the relation between the Bennett, and Braunstein-Kimble schemes. At first sight, they may seem rather different, cf. chapter 2. Though the Braunstein-Kimble scheme may also be described in terms of either wavefunctions [56, 59] or Fock-states [74], and a low-dimensional coherent state description has also been developed recently [44]. A covariant description in terms of canonically conjugate observables and their eigenstates is also possible [83], providing a description valid for both discrete and continuous dimensions.

In this section I present the description of quantum teleportation purely in the framework of Wigner-function formalism of quantum mechanics. The main emphasis is put on the case of finite dimensional Hilbert spaces, but I make some comments on the infinite dimensional limits. It will be shown that the entire process of quantum teleportation can be consistently described purely in terms of Wigner functions, and in this context, the finite and infinite dimensional cases can be treated in a conceptually uniform way.

3.4.1 Discrete Wigner functions

Consider a physical system, with states described by the N-dimensional Hilbert space $\mathscr{H}$. We define two non-commuting Hermitian operators $\hat{q}$ and $\hat{p}$ describing two canonically conjugate quantities. We will call them "position" and "momentum" respectively, though they may be realized by several physical quantities, as, for instance, photon number and Pegg-Barnett phase operators on a truncated Fock-space. The operators are defined as:

$$\hat{q} = \sum_{k=0}^{N-1} k|k\rangle\langle k|, \qquad \hat{p} = \sum_{l=0}^{N-1} l|p_l\rangle\langle p_l| \tag{3.43}$$

where the set of $|k\rangle$ position and $|p_l\rangle$ momentum eigenstates both form an orthonormal basis on $\mathscr{H}$, and

$$|p_l\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{i\frac{2\pi}{N}kl}|k\rangle \tag{3.44}$$

holds.

Wigner functions for this discrete system can be defined in a slightly different manner depending on the properties of the number N, the dimensionality of the corresponding Hilbert space. In what follows I will suppose that N is greater or equal than 3 and it is a prime number. Though it introduces some loss of generality, apart from technical details, there is no significant physical difference between the discussed case, and the remaining two possibilities. In case of $N = 2$, a different definition of the Wigner function has to be applied, while for composite N's, the phase spaces are Cartesian products of lower dimensional phase-spaces. Alternatively, one may use the formalism suggested in Ref. [51] or Ref. [61]

According to the original paper of Wootters [80], the Wigner function corresponding to a state in a Hilbert space with dimension $N \geq 3$ prime, is defined with the aid of the discrete Wigner operator

$$\hat{A}(q,p) = \sum_{r,s} \delta_{2q,r+s} e^{i\frac{2\pi}{N}p(r-s)}|r\rangle\langle s|, \tag{3.45}$$

where q and p take integer values from 0 to $N-1$. The (q,p) pairs constitute the discrete phase space. For a state described by a density matrix ρ the Wigner function is

$$W(q,p) = \frac{1}{N}\,\mathrm{tr}(\rho\hat{A}). \tag{3.46}$$

Wigner functions defined in this way obey analogous properties to those defined on infinite dimensional Hilbert spaces. The marginal distributions of the functions

$$P_q(q) = \sum_p W(q,p), \qquad P_p(p) = \sum_q W(q,p) \tag{3.47}$$

describe the statistics of measurements of observables $\hat{q}$ and $\hat{p}$ respectively.

For multipartite systems, Wigner functions are defined, similarly to the infinite dimensional case, by the expectation values of the direct product of the Wigner operators. In what follows I consider multipartite systems with Hilbert spaces of equal dimension. For a bipartite system with subsystems 1 and 2, described by the joint density matrix $\rho^{(12)}$,

$$W(q_1,p_1,q_2,p_2) = \frac{1}{N^2}\,\mathrm{Tr}(\rho^{(12)}\hat{A}_1(q_1,p_1)\otimes\hat{A}_2(q_2,p_2)) \tag{3.48}$$

Wigner functions describing a subsystem are obtained by summing the joint Wigner function over the corresponding set of the respective variables, e. g. from Eq. (3.48) one has

$$\begin{aligned} W(q_1,p_1) &= \sum_{q_2,p_2=0}^{N-1} W(q_1,p_1,q_2,p_2), \\ W(q_2,p_2) &= \sum_{q_1,p_1=0}^{N-1} W(q_1,p_1,q_2,p_2). \end{aligned} \tag{3.49}$$

For bipartite systems, the completely entangled Bell-states

$$|\Xi_{P,X}\rangle_{12} = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} e^{i\frac{2\pi}{N}kP}|k\rangle_1|k-X\rangle_2, \tag{3.50}$$

form an orthonormal basis on the $\mathscr{H}\otimes\mathscr{H}$ Hilbert space of the joint system. Here X and P stand for the *relative coordinate* and the *total momentum*. These are, up to a normalization factor, conjugate joint observables, and the Bell-states are common eigenstates of them:

$$\begin{aligned} (\hat{q}_1-\hat{q}_2)|\Xi_{p,x}\rangle_{12} &= (q_1-q_2)|\Xi_{p,x}\rangle_{12}, \\ (\hat{p}_1+\hat{p}_2)|\Xi_{p,x}\rangle_{12} &= (p_1+p_2)|\Xi_{p,x}\rangle_{12}. \end{aligned} \tag{3.51}$$

I remark that, compared to the Kimble formalism, I have introduced a canonical transformation of simultaneous position and momentum reversal in the second subsystem. This does not effect the essence of the argument, it was done in order to be consistent with the notation in Refs. [16, 37].

3.4.2 Teleportation in discrete Wigner formalism

Following Bennett [7], let us suppose that the sender, Alice, and the receiver, Bob, share the subsystems 2 and 3 in the entangled state

$$|\Xi_{0,0}\rangle_{23} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |k\rangle_2 |k\rangle_3. \tag{3.52}$$

In what follows, I shall use the term "EPR-state" for this state. The Wigner function of this state can be calculated according to Eqs. (3.45), (3.46) and (3.48), and is found to be

$$W_{\text{EPR}}(q_2, p_2, q_3, p_3) = \frac{1}{N^2} \delta_{q_2, q_3} \delta_{p_2, -p_3}. \tag{3.53}$$

Calculating the Wigner functions for subsystems 2 and 3 according to Eq. (3.49), both of them are found to be the constant $1/N^2$. From this follows that any of the marginals describe a uniform distribution. This reflects the EPR nature of the state: making observations on either of the subsystems separately, both position and momentum have random values. On the other hand, according to Eq. (3.51), some joint observables have definite values, as it is also clearly reflected by Eq. (3.53): $q_2 - q_3 = 0$ and $p_2 + p_3 = 0$. From this one may conclude that the form of the EPR Wigner function in Eq. (3.53) could even have been a plausible ansatz.

The Wigner function in Eq. (3.53) shows the connection with the EPR state used by Braunstein and Kimble for continuous variable teleportation. In the continuous variable case, for an ideal EPR state Dirac-deltas appear, corresponding to a state with infinite energy. Therefore instead of the ideal EPR state, usually two-mode squeezed vacuum is considered instead, which results in the imperfection of the protocol.

Let me consider the teleportation process. Alice, the sender and Bob the receiver have shared the EPR pair described by the Wigner function in Eq. (3.53). In addition Alice has system 1 in the arbitrary state described by a Wigner function $W_{\text{in}}(q_1, p_1)$. The joint Wigner function of the whole system is thus

$$W(q_1, p_1, q_2, p_2, q_3, p_3) = \frac{1}{N^2} W_{\text{in}}(q_1, p_1) \delta_{q_2, q_3} \delta_{p_2, -p_3} \tag{3.54}$$

Alice has to carry out a projective measurement on subsystems 1 and 2. This measurement is performed in the Bell basis which obviously projects the systems 1 and 2 on the Bell states (3.50). As it was already mentioned these states are simultaneous eigenstates of the joint observables $\hat{X}_2 = \hat{q}_1 - \hat{q}_2$ and $\hat{P}_1 = \hat{p}_1 + \hat{p}_2$. In order to describe the measurement, one has to express the Wigner function in Eq. (3.54) in terms of these variables and $X_1 = \hat{q}_1 + \hat{q}_2$ and $\hat{P}_2 = \hat{p}_1 - \hat{p}_2$, instead of q_1, p_1 and q_2, p_2. Note that because of the modulo N arithmetics, the ranges of the new variables are the same.

This canonical transformation is more straightforward in the infinite dimensional case, where one can introduce a $\frac{\sqrt{2}}{2}$ factor in the definition of the new variables, and thus it is easy to express the inverse transformation in the same fashion. In our case, a division by 2 appears in the inverse formula, which is less „elegant": the canonical transformation is accompanied by a scaling of the quantities. However, as N is odd, one may introduce a "generalized division by 2" in the modulo N sense as

$$\mathscr{D}_2(k) = \begin{cases} \frac{k}{2}, & N \text{ even} \\ \\ \frac{k+N}{2}, & N \text{ odd} \end{cases}, \tag{3.55}$$

which has the property $2\mathscr{D}_2(k) = k$. Here I emphasize again that *all* additions, subtractions and multiplications are understood in the modulo N sense. With the aid of this operation, the old variables can be expressed as

$$\begin{aligned} q_1 &= \mathscr{D}_2(X_1 + X_2), \qquad q_2 = \mathscr{D}_2(X_1 - X_2) \\ p_1 &= \mathscr{D}_2(P_1 + P_2), \qquad p_2 = \mathscr{D}_2(P_1 - P_2). \end{aligned} \tag{3.56}$$

The Wigner function in Eq. (3.54) after the transformation is

$$\begin{aligned} & W(X_1, P_1, X_2, P_2, q_3, p_3) = \frac{1}{N^2} \delta_{X_1 - X_2, 2q_3} \delta_{P_1 - P_2, -2p_3} \\ \times \quad & W_{\text{in}}(\mathscr{D}_2(X_1 + X_2), \mathscr{D}_2(P_1 + P_2)). \end{aligned} \tag{3.57}$$

At this stage, all subsystems are entangled. Note, that the canonical transformation, which is described here by introducing new variables, is physically a unitary transformation which entangles two subsystems, moreover, it cannot be carried out completely by using passive linear optical elements [55], and may require nonlinear optics [47].

Now we are ready to describe the Bell-state measurement, which results in values X_2 and P_1, the classical information, which is sent to Bob. Summing the Wigner function in Eq. (3.57) in variables X_1, P_2, q_3, p_3, one obtains the probability distribution of the measurement results, which is equal to the constant $1/N^2$. Thus we can obtain each possible measurement result with equal probability, in accordance with Bennett's description.

To describe the conditional projection by the measurement, one has to keep variables X_2 and P_1, constants, as these numbers constitute the result of the measurement, and one sums the Wigner function of Eq. (3.57) in variables X_1 and P_2, as all information about these is lost because of the projective measurement. This procedure is the exact analogue of the continuous case. The resulting Wigner function has to be renormalized, and it has the form

$$W_{\text{out}}(q_3, p_3) = W_{\text{in}}(q_3 + X_2, p_3 + P_1). \tag{3.58}$$

It is seen that the resulting Wigner function is a shifted version of the original, and the shift is determined by the result of the measurement. This is the exact analog of the continuous case. Bob, possessing the values X_2 and P_1, can restore the teleported state. The shift in a finite dimensional Hilbert space is illustrated in Fig. 3.4. Obviously, these shifts correspond to translations (canonical transformations) in a discrete phase space.

The required inverse transformation as described by Bennett is

$$U_{X_2,P_1} = \sum_k e^{i\frac{2\pi}{N}P_1 k}|k\rangle\langle k - X_2|. \tag{3.59}$$

It is easy to verify that this transformation acts on a Wigner function as

$$W'(q,p) = \langle U^{\dagger}_{X_2,P_1} A(q,p) U_{X_2,P_1}\rangle = W(q - X_2, p - P_1), \tag{3.60}$$

thus our description is perfectly consistent with Bennett's results.

The similarity of our discussion to the original description of continuous variable quantum teleportation by Braunstein and Kimble is apparent. Care should be taken however, if an infinite dimensional limit is be constructed from the description above, which is far from straightforward indeed. For instance, several nontrivial problems have to be overcome if $\hat{q}$ and $\hat{p}$ is associated with photon numbers and Pegg-Barnett phase [54, 71].

In conclusion I have shown that quantum teleportation can be described purely in terms of Wigner functions, and this could have been possible without mentioning the underlying Hilbert space. This approach has several advantages in the description of imperfections. Noisy entanglement can be treated, similarly to the continuous case, by replacing the Kronecker-deltas describing ideal entangled states with the appropriate Wigner function. While projective measurement is described by filtering with delta-functions here, a fuzzy measurement may be described by filtering with unsharp filters. This example

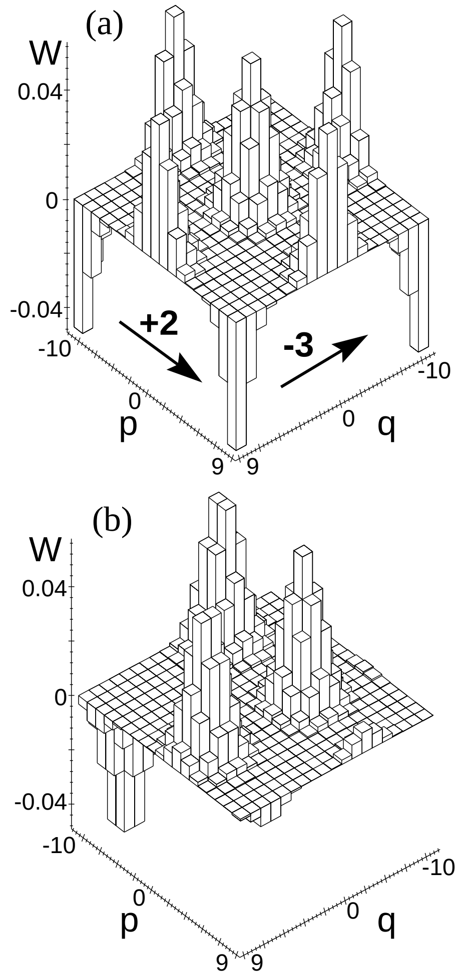

Figure 3.4: Shifting of Wigner function in a discrete phase space of a quantum system with a 19-dimensional Hilbert space. (a) shows the state, which is a discrete counterpart of the harmonic oscillator ground-state (see Ref. [58]). (b) is shifted version, according to the arrows in figure (a). Points of the phase space are indexed so that the main peak is centered in the origin of a phase space; recall the modulo N summation.

suggests that Wigner functions may prove to be a useful tool for investigating phenomena in multipartite systems with finite dimensional Hilbert spaces.

Chapter 4

Continuous variable teleportation in terms of coherent state superpositions

4.1 Introduction

This section is devoted to an alternative description of the teleportation scheme introduced by Braunstein and Kimble, presented in section 2.2.

The method developed here is based on the low-dimensional coherent state representation described in section 1.3.6. It was further generalized [43], and may become useful in the treatment of other entanglement related phenomena in the theory of continuous quantum variables, quantum cloning [16, 32, 25], or quantum dense coding [18], for instance.

In continuous variable schemes, each subsystem is a harmonic oscillator, usually a single mode light field. The measurement carried out on the light modes is quantum homodyning, described in section 1.3.9. I consider idealized homodyne detectors realizing von Neumann measurements of a given quadrature, as I do not intend to treat the effects of noises and losses on the experiment [77].

As we have seen in chapter 2, the original formulation of Braunstein and Kimble utilizes the Wigner-function formalism. In section 3.4, we have seen the connection of their scheme to that of Bennett. I intend to give a direct description of the scheme in terms of low-dimensional coherent state representations.

This chapter is organized as follows: in section 4.2, using a one-dimensional representation of quadrature eigenstates, I obtain a one-complex-plane representation of the two mode entangled states playing an important role in teleportation. In section 4.3, the description of continuous variable teleportation is provided.

4.2 Quadrature Bell-states on a coherent-state basis

The starting points of the consideration are the local measurements of a given field mode in the scheme under consideration, which are carried out by detectors measuring the value of either of the quadratures

$$\hat{q} = \frac{\hat{a}+\hat{a}^\dagger}{2}, \quad \hat{p} = \frac{\hat{a}-\hat{a}^\dagger}{2i}. \tag{4.1}$$

Throughout this chapter, I am using quadratures scaled by a $1/\sqrt{2}$ factor with respect of Eq. (1.39), for the sake of notational convenience: these scaled quadratures are the coordinates on the phase space.

According to the von Neumann projection principle, the measurement results in the projection to one of the eigenstates,

$$\hat{q}||X\rangle = X||X\rangle, \quad \hat{p}||P\rangle = P||P\rangle \tag{4.2}$$

depending on the measurement result, which is the value X or P respectively. (The symbol $||\ldots\rangle$ denotes quadrature eigenstates.)

The Bell-state detector of the teleportation scheme in argument is depicted in Figure 4.1. It consists of a $\hat{q}$-detector and a $\hat{p}$-detector, combined with a beam splitter to convert two local quadrature-measurements to a joint measurement on two modes. The whole apparatus then projects onto an entangled state of the two modes, the quadrature Bell-states, depending on the values X and P measured.

With this picture in mind, I construct the one-dimensional representation of quadrature-eigenstates. (The word dimension stands for real, and not for complex dimension througout this chapter.) Let kets containing a single number denote coherent states. I start with the following states [45]:

$$|\text{Sq. vac. p}\rangle = \mathcal{N}(r) \int_{-\infty}^{\infty} dx G_r(x)\, |x\rangle\,,$$

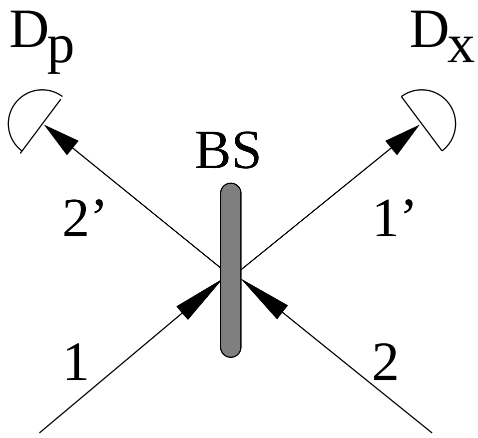

Figure 4.1: The quadrature Bell-state detector of a continuous variable teleportation scheme. Incident fields in modes 1 and 2 interfere on a lossless beam splitter. Two ideal homodyne detectors D_{q} and D_{p} measure the $\hat{q}$ quadrature in mode $1'$ and $\hat{p}$ in mode $2'$ of the resulting state.

$$|\text{Sq. vac. x}\rangle = \mathcal{N}(r) \int_{-\infty}^{\infty} dy G_r(y) \, |iy\rangle \,, \tag{4.3}$$

where

$$\mathcal{N}(r) = \frac{1}{\sqrt{\pi}} \frac{e^{r/2}}{\sqrt{e^{2r}-1}}, \quad \text{and} \quad G_r(x) = e^{-\frac{|x|^2}{e^{2r}-1}}. \tag{4.4}$$

These are superpositions of coherent states placed on the real and imaginary axis of the phase space, respectively. It is straightforward to show that the mean values of the quadratures are 0, and for their variances

$$\Delta\hat{p}^2_{|\text{Sq. vac. p}\rangle} = \Delta\hat{q}^2_{|\text{Sq. vac. x}\rangle} = \frac{e^{-2r}}{4},$$
$$\Delta\hat{q}^2_{|\text{Sq. vac. p}\rangle} = \Delta\hat{p}^2_{|\text{Sq. vac. x}\rangle} = \frac{e^{2r}}{4} \tag{4.5}$$

hold. Therefore these are the so called *squeezed vacuum states*. Squeezing means that the variance of a quadrature is decreased, while the variance of the other quadrature is increased. In Wigner-function picture this looks like as if the Wigner function of the vacuum state, the well-known Gaussian bell, was “squeezed” from a direction. If r tends towards infinity, the variance of the corresponding quadratures becomes zero, thus the states become quadrature eigenstates:

$$||P=0\rangle = \lim_{r\to\infty} |\text{Sq. vac. p}\rangle$$

$$||X=0\rangle = \lim_{r\to\infty} |\text{Sq. vac. x}\rangle\,. \tag{4.6}$$

However, $\lim_{r\to\infty} \mathscr{N}(r) = 0$, which expresses the fact that quadrature eigenstates need to be normalized in terms of probability densities instead of individual probabilities. For simplicity, in what follows I omit this normalization factor. Thus the states in Eq. (4.6) can be written as

$$\begin{aligned} ||P=0\rangle &= \lim_{r\to\infty} \int_{-\infty}^{\infty} dx G_r(x)\,|x\rangle = \int_{-\infty}^{\infty} dx\,|x\rangle \\ ||X=0\rangle &= \lim_{r\to\infty} \int_{-\infty}^{\infty} dy G_r(y)\,|iy\rangle = \int_{-\infty}^{\infty} dy\,|iy\rangle\,. \end{aligned} \tag{4.7}$$

Finally, quadrature eigenstates can be obtained by shifting states in Eq. (4.7) using the Glauber displacement operator $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$:

$$\begin{aligned} ||P\rangle &= \hat{D}(iP)||P=0\rangle = \int_{-\infty}^{\infty} dx e^{ixP}\,|x+iP\rangle \\ ||X\rangle &= \hat{D}(X)||X=0\rangle = \int_{-\infty}^{\infty} dy e^{-iXy}\,|X+iy\rangle\,. \end{aligned} \tag{4.8}$$

Now I consider the Bell-state detector. Suppose that modes 1 and 2 interfere on the lossless beam splitter BS, and then the quadratures $\hat{q}$ of mode 1 and $\hat{p}$ of mode 2 are measured, and they are found to be X and P respectively. The measurement projects the state of modes 1 and 2 at the output port of the beam splitter to

$$\left|\Psi_{\text{prod}_{X,P}}\right\rangle = ||X\rangle_1 ||P\rangle_2. \tag{4.9}$$

This is a product state basis on the Hilbert space of these two modes. The aim is to calculate the inverse beam splitter transform of the states $|\Psi_{\text{prod}_{X,P}}\rangle$, which will yield an entangled state basis. Note that the connection of displacement and entangled states have been discussed in other descriptions, too [83, 38].

Armed with the representations in Eq. (4.8), one may describe the action of a beam splitter quite simply. Two-mode coherent states interfere on beam splitters as classical fields, that is, their amplitudes transform as the annihilation operators. Particularly, one may consider a $50-50\%$ beam splitter, with phase shifts chosen so that for the output state $|\alpha\rangle_1\,|\beta\rangle_2$ the corresponding input state is $\left|(\alpha+\beta)/\sqrt{2}\right\rangle_1 \left|(\beta-\alpha)/\sqrt{2}\right\rangle_2$. Because of the linearity of the beam splitter, the inverse transform of arbitrary superpositions of coherent states may be written as:

$$\int d^2\alpha \int d^2\beta \Phi(\alpha,\beta)\, |\alpha\rangle_1 \, |\beta\rangle_2 \to \int d^2\alpha \int d^2\beta \Phi(\alpha,\beta) \left|\frac{\alpha+\beta}{\sqrt{2}}\right\rangle_1 \left|\frac{\beta-\alpha}{\sqrt{2}}\right\rangle_2 . \tag{4.10}$$

Here $\Phi(\alpha,\beta)$ is an arbitrary function, and the complex integrals may be replaced by any kinds of integrals or sums.

Eq. (4.10) can be applied in this actual case: according to the von Neumann projection principle, the state of output modes of the beam splitter is projected onto

$$|\Psi_{\mathrm{prod}_{X,P}}\rangle = ||X\rangle_1 ||P\rangle_2 = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx e^{i(xP-Xy)} \, |X+iy\rangle_1 \, |x+iP\rangle_2 , \tag{4.11}$$

thus its inverse transform, the corresponding Bell-state reads

$$|B(X,P)\rangle = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx e^{i(xP-Xy)} \left|\frac{x+iy+X+iP}{\sqrt{2}}\right\rangle_1 \left|\frac{x-iy+iP-X}{\sqrt{2}}\right\rangle_2 . \tag{4.12}$$

Introducing two complex variables

$$\gamma := \frac{x+iy}{\sqrt{2}}, \quad A := \frac{X+iP}{\sqrt{2}}, \tag{4.13}$$

the state in Eq. (4.12) reads

$$|B(X,P)\rangle = \int d^2\gamma e^{A\gamma^* - A^*\gamma} \, |\gamma + A\rangle_1 \, |\gamma^* - A^*\rangle_2 . \tag{4.14}$$

(Throughout this section, complex integrals are meant to be carried out on the whole complex plane.) These are the quadrature Bell-states playing an important role in continuous variable quantum teleportation.

Starting from the one real dimensional representation in Eq. (4.8) of the quadrature eigenstates, I have obtained a representation of a two mode state which is a superposition of two mode coherent states with amplitudes on one single complex plane.

If $A = 0$, then the following state is obtained:

$$|\Psi_{\text{EPR id.}}\rangle = \int d^2\gamma \big|\gamma\rangle_1\big\rangle \, |\gamma^*\rangle_2 . \tag{4.15}$$

It was shown in Ref. [43] that this is the one-complex plane-representation of a two-mode infinitely squeezed vacuum state. The effect of finite squeezing can be represented by the Gaussian factor $G_r(\sqrt{2}|\gamma|)$ in the integrand in (4.15), inherited from Eqs. (4.7).

The states $|\gamma\rangle_1 |\gamma^*\rangle_2$ have the remarkable property [26] of carrying more information than duplicate coherent states $|\alpha\rangle\,|\alpha\rangle$, which also makes them interesting in cloning applications [25]. I have now the quadrature Bell-states expressed as a superposition of conjugate coherent state pairs.

4.3 Continuous variable teleportation on a coherent state basis

Let us now turn our attention to the teleportation process which is depicted in Figure 4.2. Alice has an arbitrary quantum state $|\Psi_{\text{in}}\rangle$ in mode 1, which she wants to teleport to Bob. A general pure state may be written in Glauber's analytic representation as

$$\left|\Psi_{\text{in}}\right\rangle_1 = \int d^2\beta e^{-\frac{|\beta|^2}{2}} f(\beta^*)\,|\beta\rangle_1\,, \tag{4.16}$$

where $f(\beta^*)$ is an analytic function of β^*. Alice and Bob share a two-mode squeezed vacuum state

$$|\Psi_{\text{EPR}}\rangle_{23} = \int d^2\alpha G_r(\sqrt{2}|\alpha|)|\alpha^*\rangle_2|\alpha\rangle_3, \tag{4.17}$$

as EPR state for the teleportation. As discussed previously, in the ideal case, that is, infinitely squeezed and thus maximally entangled state, $G_r(\sqrt{2}|\alpha|) \to 1$. The state of the whole system of all three modes is thus initially

$$\begin{aligned}|\Psi_i\rangle_{123} = |\Psi_{\text{in}}\rangle_1 \otimes |\Psi_{\text{EPR}}\rangle_{23} = \\ \int d^2\alpha \int d^2\beta G_r(\sqrt{2}|\alpha|)e^{-\frac{|\beta|^2}{2}} f(\beta^*)|\beta\rangle_1|\alpha^*\rangle_2|\alpha\rangle_3.\end{aligned} \tag{4.18}$$

In the next step, Alice carries out a joint measurement, resulting in a pair of values X,P, which is communicated to Bob via a classical channel. The effect of this measurement is the projection of the state of modes 1 and 2 to one of the quadrature Bell-states in Eq. (4.14). Therefore, in what follows I shall omit all constant multiplying factors from

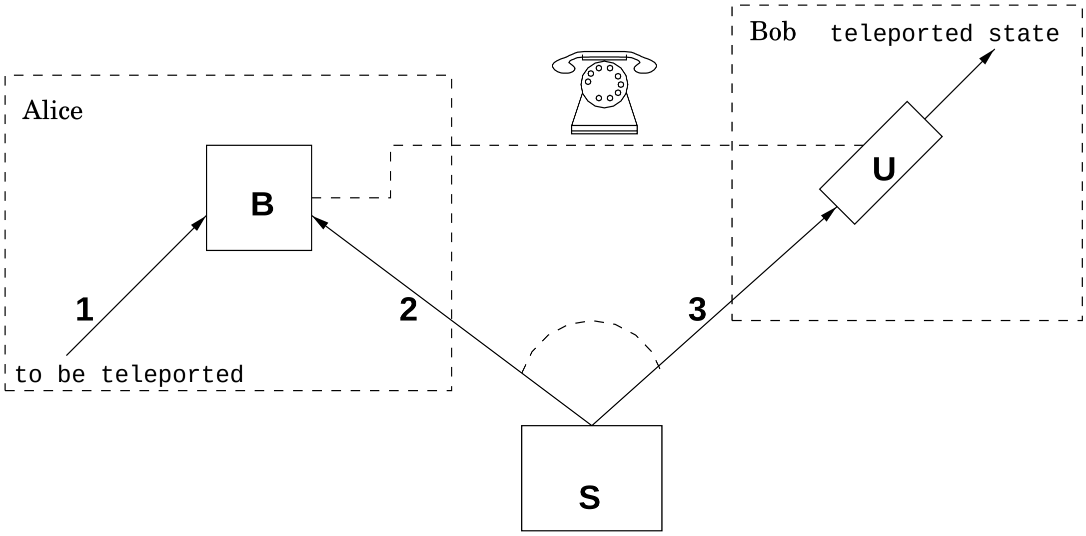

Figure 4.2: The teleportation scheme under consideration. Alice, the sender wants to teleport the state in mode 1. Alice and Bob share modes 2 and 3 in entangled state emitted by source S. Alice carries out a joint measurement on modes 1 and 2 using her box B which contains the setup in Fig. 4.1. She communicates the result to Bob via a classical channel, who carries out the appropriate unitary transformation U to restore the desired state.

the formulae. The (unnormalized) projected state of mode 3 reads

$$|\Psi_{\rm f}\rangle_3 = \ {}_{12}\langle B(X,P)|\Psi_i\rangle_{123} = \int d^2\alpha \int d^2\beta \int d^2\gamma G_r(\sqrt{2}|\alpha|) e^{-\frac{|\beta|^2}{2}} f(\beta^*) e^{A^*\gamma - A\gamma^*} \times \langle \gamma + A|\beta\rangle \langle \gamma^* - A^*|\alpha^*\rangle |\alpha\rangle_3. \tag{4.19}$$

The integrals in β and γ can be evaluated via the successive application of the Glauber's integral identity

$$\frac{1}{\pi}\int d^2\alpha e^{-|\alpha|^2+\alpha\beta^*} f(\alpha^*) = f(\beta^*), \tag{4.20}$$

which is valid for any function f analytical in α^*. Applying (4.20) twice, integrating over γ and over β, I obtain

$$|\Psi_{\rm f}\rangle_3 = \int d^2\alpha G_r(\sqrt{2}|\alpha|) e^{-\frac{|\alpha|^2}{2}} e^{-2A\alpha^*} f(\alpha^* + 2A^*)|\alpha\rangle_3. \tag{4.21}$$

In the limit of ideal entanglement (infinite squeezing), $G_r(\sqrt{2}|\alpha|) \to 1$, this state becomes

$$|\Psi_{\rm f}\rangle = D(-2A)|\Psi_{\rm in}\rangle. \tag{4.22}$$

The state in mode 3 is a shifted version of the incoming state. Bob, in the knowledge of P and X can carry out the inverse displacement to restore the original state.

Note that the displacement to be done by Bob is the identity operator if and only if $X = P = 0$, and in this case the two-mode Bell-state measured by Alice is the same as the shared entangled state. The same situation appears in the case of discrete variable teleportation.

If the entangled state is not ideal, the $G_r(\sqrt{2}|\alpha|)$ Gaussian smoothing factor appears in Eq. (4.21). After the inverse displacement, the result of the teleportation reads

$$|\Psi_{\rm f}\rangle = \int d^2\alpha G_r(\sqrt{2}|\alpha - 2A|) e^{\frac{-|\alpha|^2}{2}} f(\alpha^*)|\alpha\rangle. \tag{4.23}$$

The smoothing depends not only on r but also on A as a consequence of the finite number of photons contained in the entangled state. Let me remark that in order to calculate the fidelity of teleportation (c. f.. Ref. [38]) in the present formalism, one may average in A by forming a density matrix from the state in Eq. (4.23), and calculating the probability distribution of A from Eq. (4.21).

4.4 Summary

I have shown that using a one-complex-plane coherent state representation of quadrature states, quadrature Bell-states can be represented by integrating on a single complex plane. Using this representation I have found that an alternative and rather plausible description of continuous variable quantum teleportation can be formulated. This approach is different from all previous approaches applying Wigner functions, photon number states or quadrature-wavefunctions. Regarding the role of coherent states in the development of the theory of nonclassical states of light, my approach may prove to be useful in the further investigation of quantum teleportation and related phenomena.

Chapter 5

Optical state truncation with teleportation: few-photon interference

5.1 Few-photon interference schemes

As mentioned in section 1.3.7, nonlinear optical processes driven by pulsed pump beams generate running-wave states of the electromagnetic field containing a low number of photons. These fields can be manipulated with optical devices, such as beam splitters and phase shifters, and measurements may be carried out on them using photodetectors. I consider single mode running wave fields in this section. The first experimental realization of quantum teleportation [15], or quantum lithography [14] are typical examples of such interferometric schemes. The theoretical background of the single-mode field model and the treatment of photodetection applied here has been described in section 1.3.3. This chapter is devoted to the analysis of certain other few-photon interference arrangements.

5.1.1 "Quantum scissors" devices

In a paper of Pegg, Phillips and Barnett [62] it is shown that a one-mode traveling wave optical state can be truncated so as to leave only its vacuum and one-photon components. This can be regarded as a tool for quantum state design. The proposed arrangement, called "quantum scissors", consists of two beam splitters and two photon-counting detectors. It exploits quantum measurement and nonlocality. As shown by Villas-Bôas, de Almeida, and Moussa [76], it is also a realization of quantum teleportation of a two state system, the basis states being the vacuum and the one-photon Fock state. The above authors analyze the operation of the arrangement in a noisy environment. It has also been shown [4] that quantum scissors have good fidelity also in presence of imperfections. The quantum scissors device is interesting at least from two points of view: quantum state design and teleportation of states of traveling wave electromagnetic field.

Quantum scissors may be capable of converting a classical state to a highly nonclassical one. For example, if the input is a low intensity coherent state, the truncation yields a coherent superposition of vacuum and one-photon state. This state is known to possess squeezing properties [79], and can be used as a reference state in the projection synthesis technique[3, 69]. The quantum scissors work for other (even mixed) input states too, thus they are capable of generating several kinds of superpositions and mixtures of vacuum and one-photon states. The question arises naturally, whether the class of preparable states can be enlarged. One possibility is described by Dakna et al. [29], which applies more beam splitters and detectors. I follow a different way: I do not raise the number of components of the arrangement, but examine the facilities introduced by the freedom of using beam splitters with appropriate parameters. It turns out that the truncation so as to leave vacuum, one-, and two-photon superpositions needs no significant extension of current experimental expertise. This generalized quantum scissors device can generate a larger class of nonclassical states. For example, cutting a squeezed vacuum state, a coherent superposition of vacuum and two-photon states can be obtained, which may also be used as a reference state in projection synthesis; its squeezing properties have been analyzed in Ref. [79]. The truncation of coherent states also makes an interesting class of nonclassical states feasible. The arrangement works for any pure and mixed input state.

The other aspect of the operation of the quantum scissors device is quantum teleportation. Generalized quantum scissors create a superposition of vacuum, one- and two-photon Fock states of a one-mode traveling wave field, and teleport it at the same time. If the input state of generalized scissors is already a superposition of this kind, it is simply teleported. This is a teleportation on the three-dimensional Hilbert space spanned by $\{|0\rangle, |1\rangle, |2\rangle\}$. I analyze this particular situation in detail. The discussion yields a suggestive insight into the process of transporting quantum information in this case, which also led me to the generalization of the $SU(2)$ beam-splitter theory to six-ports.

I have also investigated the possibility of further generalization: truncating up to the n-th Fock component.

In what follows, in section 5.1.2 I introduce generalized quantum scissors devices: I investigate the possibility of truncation up to the n photon components. Then I analyze the teleportation in detail, employing the $SU(2)$ formalism described in section 1.3.8. The idea of $SU(3)$ tritter theory, which was suggested by the latter consideration will be described in section 5.2.

5.1.2 State truncation up to two-photon states

The quantum scissors device is depicted in Fig. 5.1. The notation used is also shown in the figure. I label the modes with their annihilation operators. We are given an arbitrary state in the input mode $\hat{a}_3$. For simplicity consider a pure state

$$|\Psi_{\text{in}}\rangle = \sum_{k=0}^{\infty} \gamma_k |k\rangle \tag{5.1}$$

as input, but the generalization to mixed states is straightforward. The aim is to obtain a truncated state

$$|\Psi_{\text{out}}\rangle = \sqrt{N}(\gamma_0|0\rangle + \gamma_1|1\rangle + \gamma_2|2\rangle). \tag{5.2}$$

as output in mode $\hat{c}_1$, $N = 1/\sum_{k=0}^{2} |\gamma_k|^2$ being a renormalization constant. The operation of the device consists of unitary evolution and a measurement process. The unitary evolution can be divided into two steps: action of the BS_1 and BS_2 beam splitters. Operators $\hat{a}$, $\hat{b}$, and $\hat{c}$ belong to the stages of unitary evolution, and their indices refer to the spatial modes. At the beginning, modes $\hat{a}_1$ and $\hat{a}_2$ are in a given state $|\Psi_{12}\rangle$. The state of the whole system of the three modes is $|\Psi\rangle = |\Psi_{12}\rangle \otimes |\Psi_{\text{in}}\rangle$ initially.

I describe the unitary evolution in the Heisenberg picture. In the first step the effect of BS_1 can be described by the unitary transformation

$$\begin{pmatrix} \hat{b}_1^\dagger \\ \hat{b}_2^\dagger \end{pmatrix} = \begin{pmatrix} e^{-i\varphi_t}\cos\tau_1 & e^{-i\varphi_r}\sin\tau_1 \\ -e^{i\varphi_r}\sin\tau_1 & e^{i\varphi_t}\cos\tau_1 \end{pmatrix} \begin{pmatrix} \hat{a}_1^\dagger \\ \hat{a}_2^\dagger \end{pmatrix}, \tag{5.3}$$

while the input mode is not modified: $\hat{a}_3^\dagger = \hat{b}_3^\dagger$. In Eq. (5.3), φ_t and φ_r are the phase shifts imparted to the transmitted and reflected beams, $(\cos\tau_1)^2$ and $(\sin\tau_1)^2$ are the transmittance and reflectance of the beam splitter respectively. The next step is the action of BS_2

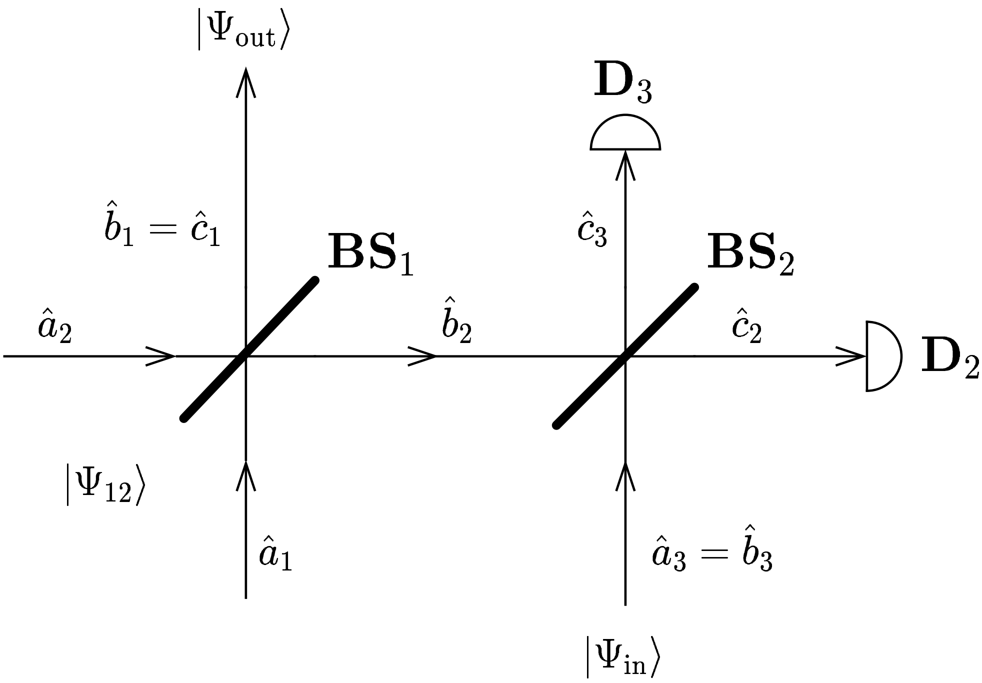

Figure 5.1: The "quantum scissors" device, the subject of the analysis. It consists of the BS_1 and BS_2 beam splitters, and D_2, D_3 photon counters. The numbering of detectors is consistent with the indexing of spatial modes . $|\Psi_{in}\rangle$ is the incoming state, $|\Psi_{out}\rangle$ is the output state. The modes $\hat{a}_1$ and $\hat{a}_2$ are in the $|\Psi_{12}\rangle$ ancillary state necessary for the operation of the device. The annihilation operators of the spatial modes are indicated.

on modes $\hat{b}_2$ and $\hat{b}_3$ yielding $\hat{c}_2$ and $\hat{c}_3$,

$$\begin{pmatrix} \hat{c}_2^\dagger \\ \hat{c}_3^\dagger \end{pmatrix} = \begin{pmatrix} e^{-i\eta_t}\cos\tau_2 & e^{-i\eta_r}\sin\tau_2 \\ -e^{i\eta_r}\sin\tau_2 & e^{i\eta_t}\cos\tau_2 \end{pmatrix} \begin{pmatrix} \hat{b}_2^\dagger \\ \hat{b}_3^\dagger \end{pmatrix}, \tag{5.4}$$

while $\hat{c}_1 = \hat{b}_1$. This is followed by a measurement, i.e. a projection to a photon number eigenstate in modes $\hat{c}_2$ and $\hat{c}_3$. The detectors are assumed to be ideal photon counters. Given a reference state $|\Psi_{12}\rangle$ one can find suitable parameters for BS_1 and BS_2 to carry out the state truncation described above.

The ancillary state $|\Psi_{12}\rangle$ has to be experimentally available in order to make the idea of quantum scissors realistic. The desired output state of Eq. (5.2) should contain at most two photons. These two photons originate from $|\Psi_{12}\rangle$, since evidently no light reaches the output from the input. Thus $|\Psi_{12}\rangle$ has to contain at least two photons. It is convenient to choose the state $|\Psi_{12}\rangle = |11\rangle$. This state, consisting of a pair of temporally correlated photons, can be generated directly using the prevalent technique of nondegenerate parametric down-conversion. Other two-mode states possessing two photons can also be obtained from this state with an extra beam splitter, which would be a redundant element in the scheme.

Suppose that from $|\Psi_{12}\rangle = |11\rangle$ the BS_1 general beam splitter produces the intermediate state

$$|\Psi'_{12}\rangle = \beta_0|20\rangle + \beta_1|11\rangle + \beta_2|02\rangle, \tag{5.5}$$

so the state of all the three modes after the first step of unitary evolution is the product of this and the state in (5.1):

$$\begin{aligned} |\Psi'\rangle = \quad & |\Psi'_{12}\rangle \otimes |\Psi_{\mathrm{in}}\rangle = \sum_{n=0}^{\infty} \gamma_n (\beta_0|20n\rangle + \beta_1|11n\rangle + \beta_2|02n\rangle) \\ = \quad & \sum_{n=0}^{\infty} \frac{\gamma_n}{\sqrt{n!}} \left(\frac{\beta_0}{\sqrt{2}} \hat{b}_1^{\dagger 2} \hat{b}_3^{\dagger n} + \beta_1 \hat{b}_1^\dagger \hat{b}_2^\dagger \hat{b}_3^{\dagger n} + \frac{\beta_2}{\sqrt{2}} \hat{b}_2^{\dagger 2} \hat{b}_3^{\dagger n} \right) |000\rangle. \end{aligned} \tag{5.6}$$

As the result of the action of BS_2, $|\Psi'\rangle$ turns into

$$\begin{aligned} |\Psi''\rangle = \quad & \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} \sum_{k=0}^{n} \binom{n}{k} (\sin\tau_2)^k (\cos\tau_2)^{n-k} e^{i(k\eta_r - (n-k)\eta_t)} \\ \times\Big(\quad & \frac{\beta_0}{\sqrt{2}} \hat{c}_1^{\dagger 2} \hat{c}_2^{\dagger k} \hat{c}_3^{\dagger n-k} + \beta_1 \cos\tau_2 e^{i\eta_t} \hat{c}_1^\dagger \hat{c}_2^{\dagger k+1} \hat{c}_3^{\dagger n-k} - \beta_1 \sin\tau_2 e^{-i\eta_r} \hat{c}_1^\dagger \hat{c}_2^{\dagger k} \hat{c}_3^{\dagger n-k+1} \\ + \quad & \frac{\beta_2}{\sqrt{2}} (\cos\tau_2)^2 e^{2i\eta_t} \hat{c}_2^{\dagger k+2} \hat{c}_3^{\dagger n-k} + \frac{\beta_2}{\sqrt{2}} (\sin\tau_2)^2 e^{2i\eta_r} \hat{c}_2^{\dagger k} \hat{c}_3^{\dagger n-k+2} \end{aligned}$$

$$- \quad 2\frac{\beta_2}{\sqrt{2}}\cos\tau_2\sin\tau_2 e^{i(\eta_t-\eta_r)}\hat{c}_2^{\dagger k+1}\hat{c}_3^{\dagger n-k+1}\Big)|000\rangle. \tag{5.7}$$

The output state in the case of a given detection event can now be determined by projecting $|\Psi''\rangle$ to the number state corresponding to the result of the measurement carried out on modes $\hat{c}_2$ and $\hat{c}_3$. Due to considerations on photon number conservation it suffices to examine the detection events in which the total number of detected photons is two. Let us examine the case in which one-photon on D_2 and one on D_3 detectors are detected in coincidence. It will turn out that the other two possible detection events (two photons on one of the detectors and no photons on the other) are inadequate choices.

After the coincident detection of one photon on D_2 and one on D_3 the state of the system becomes the projection of $|\Psi''\rangle$ of Eq. (5.7) to $|11\rangle$ in modes 2 and 3. The state of the output mode obtained this way reads up to a normalization constant:

$$-\sqrt{2}\cos\tau_2\sin\tau_2 e^{i\eta}\beta_2\gamma_0|0\rangle + \cos(2\tau_2)\beta_1\gamma_1|1\rangle + \sqrt{2}\cos\tau_2\sin\tau_2 e^{-i\eta}\beta_0\gamma_2|2\rangle, \tag{5.8}$$

where $\eta = \eta_t - \eta_r$. Comparing to the desired state in Eq. (5.2), to achieve the truncation

$$-\sqrt{2}\cos\tau_2\sin\tau_2 e^{i\eta}\beta_2 = \sqrt{2}\cos\tau_2\sin\tau_2 e^{-i\eta}\beta_0 = \cos(2\tau_2)\beta_1 = K \tag{5.9}$$

must hold. This is the condition for the β coefficients of Eq. (5.5). The efficiency of the truncation is K^2/N, where N is the renormalization constant of Eq. (5.2). It is maximal, if K is maximal (this depends on the device), and $N = 1$ (this depends on the input state). The condition for the beam splitter parameters to be chosen optimally is that K has to be maximal.

Now, given the state $|\Psi_{12}\rangle = |11\rangle$ incident on BS_1 a set of parameters for BS_1 and BS_2 has to be found to make BS_1 capable of generating the state in Eq. (5.5) with the β coefficients fulfilling Eq. (5.9). The intermediate state $|\Psi'_{12}\rangle$ in Eq. (5.5), leaving the beam splitter, is a point of the vector space spanned by the vectors $\{|20\rangle, |11\rangle, |02\rangle\}$. Varying the parameters φ_t, φ_r, and τ_1 of BS_1, this point perambulates a set of points in this vector space. (This is called $SU(2)$-orbit of the point $|11\rangle$.) The coordinates of the parametrized set of points read

$$\cos(2\tau_1)|11\rangle + \frac{\sqrt{2}}{2}\sin(2\tau_1)e^{i\varphi}|20\rangle - \frac{\sqrt{2}}{2}\sin(2\tau_1)e^{-i\varphi}|02\rangle, \tag{5.10}$$

where $\varphi = \varphi_t - \varphi_r$. Equation (5.9) also defines a parametrized set of points in this vector space with coordinates β_1, β_2, and β_3, the parameters are η and τ_2. Each point in this set represents the appropriate state that is required for state truncation if the parameters of BS_2 are chosen to be η and τ_2. The required beam splitter parameters are the coordinates of the intersection of the two point-sets in Eqs. (5.9) and (5.10).

The solution is the following: for the phase shifts $\varphi_t - \varphi_r = \eta_t - \eta_r$ must hold. Otherwise the relative phase of the Fock components gets modified. One may choose $\varphi_t = \varphi_r = \eta_t = \eta_r = 0$, which is convenient, because in this case the matrices in Eqs. (5.3) and (5.4) are real. The τ parameters have to satisfy

$$\tan(2\tau_1)\tan(2\tau_2) = 2. \tag{5.11}$$

The factor K in Eq. (5.9), and thus the maximum probability of the detection of the coincidence in argument, depends on the τ values. K itself also has a maximum at the optimal choice of τ parameters

$$\tau_1 = \tau_2 = \frac{1}{2}\arctan(\pm\sqrt{2}). \tag{5.12}$$

Thus in the optimal case the two beam splitters have to be identical, with transmittance either 0.21, or 0.79. The optimum value of the renormalization factor is $K = 1/3$, which means that the coincident detection of one photon on each detector occurs at most in $1/9$ of the cases. In these cases the state truncation is successful. The probability is exactly $1/9$ if the incoming state is already a superposition of vacuum, one, and two-photon states. This is the case of teleportation and the teleportation efficiency is $1/9$. Otherwise the probability is proportional to $1/N = \sum_{n=0}^{2} |\gamma_n|^2$. The device can be used with success if N has a low value, i.e. the incoming state contains the vacuum, one-, and two-photon states with large weights.

Let me return to the case of detecting two photons on one of the detectors and none on the other. These are interesting counter-examples, since the above discussed intersection of the point-sets are empty in these cases, thus the state truncation cannot be carried out using any kind of beam splitters.

5.1.3 Further generalization of quantum scissors

One may investigate the possibilities for further generalization of the quantum scissors device using the same arrangement. The aim is now to truncate the number-state expansion of an arbitrary incoming state (Eq. (5.1)) up to the n-photon component. In what follows I take a more general point of view: I omit the beam splitter BS_1, and suppose that the intermediate state

$$|\Psi'_{12}\rangle = \sum_{k=0}^{n} \beta_k |n-k,k\rangle, \tag{5.13}$$

which appeared after the operation of BS_1 up till now, is already prepared with some method. I don't pay attention to the preparation of this state, though it may be generated by BS_1 using some reference state $|\Psi_{12}\rangle$. Furthermore, suppose that the D_2 and D_3 are ideal photon counters and they are counting d_2 and d_3 photons respectively. The method is the same as in the case of the truncation up to two-photon components: the state of the system after the action of BS_2 has to be calculated, and the result has to be projected to the appropriate state determined by the measurement result. The difference is that since BS_1 is omitted, the result will be obtained in terms of a set of β parameters of Eq. (5.13) and parameters of BS_2.

In general, the state obtained reads

$$|\Psi_{\text{out}}\rangle = \frac{\sqrt{N}}{K} \sum_{j=0}^{\infty} \sum_{k=0}^{n} \beta_k \gamma_j D_{jk} |k\rangle = \sqrt{N} \sum_{k=0}^{n} \gamma_k |k\rangle, \tag{5.14}$$

where $N = 1/\sum_{k=0}^{n} |\gamma_k|^2$ and K are normalization constants and D_{jk} describes both BS_2 and the measurement carried out. It is easy to show that Eq. (5.14) can be solved for the β coefficients if and only if $d_2 + d_3 = n$. That is, a total photon number of n is detected. (Both $\hat{c}_2$ and $\hat{c}_3$ are assumed to be in photon number eigenstates.)

Just as in the case of truncation up to the two photon component, the desired measurement occurs with a probability of K^2/N. K^2 describes the efficiency of the truncation in function of parameter τ_2 of BS_2. An optimal quantum scissors device can be obtained by choosing τ_2 so that K^2 is maximal.

I will discuss two detection events as examples. If D_2 detects n photons and D_3 none, Eq. (5.14) has the solution

$$\beta_k = \frac{K e^{-i(k\eta_t + (n-k)\eta_r)}}{\sqrt{\binom{n}{k} (\cos\tau_2)^k (\sin\tau_2)^{n-k}}}, \tag{5.15}$$

and

$$K^2 = \left(\sum_{k=0}^{n} \frac{1}{\binom{n}{k} (\cos\tau_2)^{2k} (\sin\tau_2)^{2n-2k}} \right)^{-1}. \tag{5.16}$$

From Eq. 5.16 it can be seen that the optimal value of the $\cos\tau_2$ transmittance is $1/2$, BS_2 has to be a 50–50% beam splitter. For $n = 1$ it is the Pegg-Phillips-Barnett device. For $n = 2$, the efficiency of the process would be approximately the same as that for the case discussed in 5.1.2, but the required $|\Psi_{12}\rangle$ state of Eq. 5.15 cannot be prepared from the state $|11\rangle$ using an additional beam splitter (BS_1 in section 5.1.2).

The other example is, if D_2 detects $n-1$ photons and D_3 one photon at the same time, Eq. (5.14) gives

$$\beta_k = \frac{K e^{-i((k-1)\eta_t + (n-k-1)\eta_r)}}{\sqrt{\frac{1}{n}\binom{n}{k}} (\cos\tau_2)^{k-1} (\sin\tau_2)^{n-k-1} [n(\cos\tau_2)^2 - k]}, \tag{5.17}$$

and

$$K^2 = \left(\sum_{k=0}^{n} \frac{n}{\binom{n}{k} (\cos\tau_2)^{2k-2} (\sin\tau_2)^{2n-2k-2} [n(\cos\tau_2)^2 - k]^2} \right)^{-1}. \tag{5.18}$$

For $n = 2$ this is the case discussed in section 5.1.2.

Table 5.1 contains the optimal transmittance $(\cos^2\tau_2)$ of BS_2 and the maximal efficiency values for the two examples above. Note the significant decrease in efficiency for large photon numbers.

5.1.4 The teleportation aspect of quantum scissors in terms of $SU(2)$ symmetry

Consider a quantum scissors device for cutting up to the two-photon component, as described in section 5.1.2. If the input state is already some superposition of states with maximum two photons:

$$|\Psi_{\text{in}}\rangle = A_0|0\rangle_3 + A_1|1\rangle_3 + A_2|2\rangle_3, \tag{5.19}$$

the quantum scissors device is simply a teleporter. My aim is now to show in detail, how it works.

Again, I consider a single Beam-splitter BS_2 from the scheme, and the state

$$|\Psi_{\text{EPR}}\rangle = C_{-1}|2\rangle_1|0\rangle_2 + C_0|1\rangle_1|1\rangle_2 + C_1|0\rangle_1|2\rangle_2 \tag{5.20}$$

n	$d_2 = n,$ $\cos^2 \tau_2$	$d_3 = 0$ K^2	$d_2 = n-1,$ $\cos^2 \tau_2$	$d_3 = 1$ K^2
1	0.5	0.25	0.5	0.25
2	0.5	0.10	0.21 or 0.79	0.11
3	0.5	0.047	0.5	0.047
4	0.5	0.023	0.38 or 0.62	0.028
5	0.5	0.012	0.5	0.019
6	0.5	0.0062	0.42 or 0.58	0.012
7	0.5	0.0032	0.5	0.0093
8	0.5	0.0016	0.44 or 0.56	0.0056

Table 5.1: The optimal transmittance $(\cos^2 \tau_2)$ of the beam splitter BS_2 and the maximal efficiency (K^2) of the generalized quantum scissors device cutting up to the n-photon Fock components. Photon counters D_2 and D_3 count d_2 and d_3 photons respectively. Many of the entries belong to hypothetical arrangements that are not experimentally feasible at the present state of art.

in modes $\hat{b}_1$ and $\hat{b}_2$ is supposed to be prepared already by some method. (I have applied $SU(2)$ notation in the indices of C coefficients). Modes $\hat{b}_2$ and $\hat{b}_3$ interfere on the beam splitter BS. The output ports of the beam splitter are incident on detectors, which are supposed to be ideal photon counters realizing projective quantum measurement on the number state basis. The teleportation is successful, if the detectors count one single photon each, in coincidence.

The measurement, i.e. the coincident detection of a single photon on the each of the detectors after the beam splitter, is the annihilation of two photons to vacuum, described by the operator $\hat{c}_2\hat{c}_3$. Therefore, in order to obtain the teleported state, I write the state of all three modes $|\Psi_\mathrm{m}\rangle_{123}$ after the interaction on the beam splitter into the form

$$|\Psi_\mathrm{m}\rangle_{123} = \hat{A}^\dagger(\hat{c}_2^\dagger, \hat{c}_3^\dagger, \hat{b}_1^\dagger)|0\rangle, \tag{5.21}$$

where the operator $\hat{A}^\dagger(\hat{c}_2^\dagger, \hat{c}_3^\dagger, \hat{b}_1^\dagger)$ is a polynomial of the creation operators, and generates $|\Psi_m\rangle_{123}$ from the vacuum. This will be exactly Eq. (5.7), with our novel notation. The projection by the measurement drops all the summands in the expression of $\hat{A}^\dagger(\hat{c}_2^\dagger, \hat{c}_3^\dagger, \hat{b}_1^\dagger)$, except for that containing $\hat{c}_2^\dagger \hat{c}_3^\dagger$, thus the resulting (teleported) state can be read out from the expression of $\hat{A}^\dagger(\hat{c}_2^\dagger, \hat{c}_3^\dagger, \hat{b}_1^\dagger)$. As $\hat{c}_2$ and $\hat{c}_3$ originate from a beam splitter transformation, it is worth collecting them into $SU(2)$ multiplets. Let us introduce the notation

$$^{2l}\hat{M}_{l_3} = \hat{c}_2^{\dagger l+l_3} \hat{c}_3^{\dagger l-l_3}, \quad l_3 = -l \dots l \tag{5.22}$$

for the outcome operators. These are groups of creation operators indexed by the $SU(2)$ indices. Furthermore, given a set of arbitrary coefficients $^{2l}\mathscr{A}_{l_3},\ l_3 = -l \dots l$, let there be

$$^{2l}\hat{\mathscr{M}}_{\mathscr{A}} = \sum_{l_3=-l}^{l} {}^{2l}\mathscr{A}_{l_3} {}^{2l}\hat{M}_{l_3} \tag{5.23}$$

a linear combination of outcome operators in the l-th multiplet, with coefficients $^{2l}\mathscr{A}_{l_3}$. Different calligraphic letters in the index shall mean a different set of parameters in this notation.

Notice that maximum of 4 photons can be present at the beam splitter. The coefficients A_0, A_1, A_2 from Eq. (5.19) should appear in $\hat{A}^\dagger(\hat{c}_2^\dagger, \hat{c}_3^\dagger, \hat{b}_1^\dagger)$ before $\hat{b}_1^\dagger$ on powers determined by Eq. (5.19), due to the linearity of the system. Thus in general we have

$$\begin{aligned} \hat{A}^\dagger(\hat{c}_2^\dagger, \hat{c}_3^\dagger, \hat{b}_1^\dagger) = \quad & A_0({}^2\hat{\mathscr{M}}_{\mathscr{A}} + {}^1\hat{\mathscr{M}}_{\mathscr{B}}\hat{b}_1^\dagger + {}^0\hat{\mathscr{M}}_{\mathscr{C}}\hat{b}_1^{\dagger 2}) \\ + \quad & A_1({}^3\hat{\mathscr{M}}_{\mathscr{D}} + {}^2\hat{\mathscr{M}}_{\mathscr{E}}\hat{b}_1^\dagger + {}^1\hat{\mathscr{M}}_{\mathscr{F}}\hat{b}_1^{\dagger 2}) \\ + \quad & \frac{A_2}{\sqrt{2}}({}^4\hat{\mathscr{M}}_{\mathscr{G}} + {}^3\hat{\mathscr{M}}_{\mathscr{H}}\hat{b}_1^\dagger + {}^2\hat{\mathscr{M}}_{\mathscr{I}}\hat{b}_1^{\dagger 2}). \end{aligned} \tag{5.24}$$

This is a more instructive form of Eq. (5.7). The coefficients C of the entangled state in Eq. (5.20), and the parameters of the unitary operator describing the beam splitter BS_2 are included in the coefficients denoted by calligraphic letters.

It can be noticed that the multiplet structure suggested by the nature of the beam splitter transformation is reflected in the structure of the operator creating the output state. Only the outcomes in the $^2\hat{\mathscr{M}}_{\mathscr{A}}, {}^2\hat{\mathscr{M}}_{\mathscr{E}}, {}^2\hat{\mathscr{M}}_{\mathscr{I}}$ multiplets appear with all three A coefficients of the input state. Only the outcomes in these multiplets can provide teleportation, since the state obtained after the measurement on mode $\hat{b}_3$ depends on all three A coefficients. In case of a measurement outcome corresponding to an other multiplet some of the information is lost. The whole information is transferred if the total number of detected photons is 2.

The ${}^{2l}\mathscr{A}$, ${}^{2l}\mathscr{E}$, and ${}^{2l}\mathscr{I}$ coefficients depend on the beam splitter parameters and the C parameters of the entangled state in Eq. (5.20). It is possible to set these parameters so that ${}^{2}\mathscr{A}_0 = {}^{2}\mathscr{E}_0 = {}^{2}\mathscr{I}_0 = 1/3$. The actual determination of the appropriate entangled state and beam splitter is a geometrical problem, which I have discussed in the previous section. In case of measurement outcome described by ${}^{2}\hat{M}_0$, i.e. detection of one photon on both detectors in coincidence, causes the output in mode 1 to become the same as the input state in Eq. (5.19) was. This is the case of successful teleportation, which happens in the $1/9$ of the cases, regardless of the input state in Eq. (5.19).

In this section I have demonstrated on an example that the application of multiplet concept in the description of photon number conservation can be indeed useful in understanding operation of few-photon interference devices.

5.2 Outline of an $SU(3)$ theory of tritters

The question naturally arises whether one can treat a passive linear optical six-port or tritter in a similar manner to beam-splitters. Such devices can be realized either as a set of three coupled waveguides or as a combination of beam-splitters and phase-shifters[66]. There are now three input and three output modes, thus both the input and output can be regarded as three-dimensional oscillators. Three-dimensional oscillators are well known to possess $SU(3)$ symmetry. On the other hand, the tritter can be described, similarly to the beam-splitter (1.73), by a unitary operator which is now element of $SU(3)$:

$$\hat{b}_i = \sum_{j=1}^{3} U_{ij}\hat{a}_j, \quad i = 1,2,3 \quad U \in SU(3), \tag{5.25}$$

where $\hat{a}_i$-s and $\hat{b}_i$-s are the annihilation operators for the input and output modes respectively. Thus we can follow the similar way, as in the case of beam splitters: first describing the bosonic realization of $\mathfrak{su}(3)$ algebra and the structure of multiplets, then introducing some details of the tritter-transformation.

The lowest dimensional faithful representation of $\mathfrak{su}(3)$ algebra consists of eight 3×3

matrices $\frac{1}{2}\hat{\lambda}_i,\ i = 1\dots 8$, where λ_i-s are the Gell-Mann matrices, explicitly:

$$\hat{\lambda}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \hat{\lambda}_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\hat{\lambda}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \hat{\lambda}_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$\hat{\lambda}_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad \hat{\lambda}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\hat{\lambda}_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \hat{\lambda}_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{5.26}$$

The bosonic realization is given in a similar form as in Eq. (1.75) in the $\mathfrak{su}(2)$ case, namely for the input operators:

$$\hat{F}_i = \frac{1}{2} \left(\hat{a}_1^\dagger \quad \hat{a}_2^\dagger \quad \hat{a}_3^\dagger\right) \begin{pmatrix} & & \\ & \hat{\lambda}_i & \\ & & \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{pmatrix}, \tag{5.27}$$

whereas the realization $\hat{G}_i,\ i = 1\dots 8$ for the output field is defined in the same way with the operators $\hat{b}_k,\ k = 1\dots 3$.

Let us describe the multiplet structure at the input port. In order to do so, I introduce some operators usually applied in this context:

$$\begin{aligned} T_\pm = F_1 \pm iF_2,\ U_\pm = F_6 \pm iF_7, \\ V_\pm = F_4 \pm iF_5,\ T_3 = F_3, Y = \frac{2}{\sqrt{3}} F_8. \end{aligned} \tag{5.28}$$

The eigenvalues of the two commuting operators T_3 and Y are applied for labelling of the multiplets (Such as the third component of angular momentum in the $SU(2)$ case). The others appear to be "ladder-operators", which allow "movements" in the multiplets. Before going into details, let us give the explicit form of all the operators for the input field. The generators are

$$\hat{F}_1 = \frac{1}{2}\left(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1\right)$$

$$\begin{aligned}
\hat{F}_2 &= \frac{i}{2}\left(\hat{a}_2^\dagger\hat{a}_1 - \hat{a}_1^\dagger\hat{a}_2\right)\\
\hat{F}_3 &= \frac{1}{2}\left(\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2\right)\\
\hat{F}_4 &= \frac{1}{2}\left(\hat{a}_1^\dagger\hat{a}_3 + \hat{a}_3^\dagger\hat{a}_1\right)\\
\hat{F}_5 &= \frac{i}{2}\left(\hat{a}_3^\dagger\hat{a}_1 - \hat{a}_1^\dagger\hat{a}_3\right)\\
\hat{F}_6 &= \frac{1}{2}\left(\hat{a}_2^\dagger\hat{a}_3 + \hat{a}_3^\dagger\hat{a}_2\right)\\
\hat{F}_7 &= \frac{i}{2}\left(\hat{a}_3^\dagger\hat{a}_2 - \hat{a}_2^\dagger\hat{a}_3\right)\\
\hat{F}_8 &= \frac{1}{2\sqrt{3}}\left(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2 - 2\hat{a}_3^\dagger\hat{a}_3\right),
\end{aligned} \tag{5.29}$$

and the other operators:

$$\begin{aligned}
\hat{T}_+ &= \hat{a}_1^\dagger\hat{a}_2\\
\hat{T}_- &= \hat{a}_2^\dagger\hat{a}_1\\
\hat{U}_+ &= \hat{a}_2^\dagger\hat{a}_3\\
\hat{U}_- &= \hat{a}_3^\dagger\hat{a}_2\\
\hat{V}_+ &= \hat{a}_1^\dagger\hat{a}_3\\
\hat{V}_- &= \hat{a}_3^\dagger\hat{a}_1\\
\hat{T}_3 &= \frac{1}{2}\left(\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2\right)\\
\hat{Y} &= \frac{1}{3}\left(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2 - 2\hat{a}_3^\dagger\hat{a}_3\right).
\end{aligned} \tag{5.30}$$

Note that according to the ladder operators, the presence of the three $\mathfrak{su}(2)$ subalgebras is apparent. Let us now examine the structure of the multiplets. There are two Casimir operators of $SU(3)$, but they have a rather difficult structure, thus it is conventional to use some other indexing of the multiplets.

In order to index states corresponding to the same multiplet, the eigenvalues of the $\hat{T}_3$ and $\hat{Y}$ operator appear to be suitable, which are linear combinations of the number operators $\hat{a}_1^\dagger\hat{a}_1$ $\hat{a}_2^\dagger\hat{a}_2$ and $\hat{a}_3^\dagger\hat{a}_3$, and they commute with them. Thus the eigenstates of these number-operators, the $|nlm\rangle$ Fock-states, are the eigenstates of $\hat{T}_3$ and $\hat{Y}$, with the eigenvalues.

$$\hat{T}_3|nlm\rangle = \frac{1}{2}(n-l)|nlm\rangle, \quad \hat{Y}|nlm\rangle = \frac{1}{3}(n+l-2m)|nlm\rangle. \tag{5.31}$$

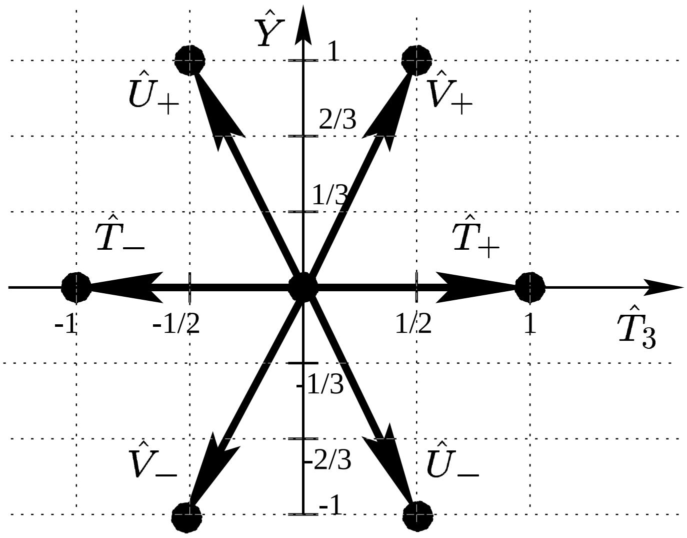

Figure 5.2: Action of the ladder operators on the $T_3 - Y$ plane

Thus the multiplets can be visualized in the T_3–Y plane.

If one of the elements of a given multiplet is known, the others can be constructed using the $\hat{U}_\pm, \hat{V}_\pm, \hat{T}_\pm$ ladder-operators. The action of these operators is shown in Fig. 5.2. It can be seen that the $\mathfrak{su}(3)$ algebra contains three $\mathfrak{su}(2)$ subalgebras symmetrically.

Generally, SU(3) multiplets are hexagon (truncated triangle) shaped on $T_3 - Y$ plane. (For a simple explanation see e. g. in Ref. [53]). Due to the difficult structure of Casimir operators (not detailed here), usually the dimensions of these polygons are used for indexing the states. The multiplet denoted by (λ, μ) has

$$\lambda = 2T_3, \text{ at } Y = Y_{\text{max}} \tag{5.32}$$

$$\mu = 2T_3, \text{ at } Y = Y_{\text{min}}. \tag{5.33}$$

Due to the additional symmetry of interchanging of bosons however, only the multiplets $(n, 0)$ *can be realized, where n denotes the total number of the photons.* Some of the multiplets are visualized on the $T_3 - Y$ plane in Fig. 5.3.

Having described the multiplets, now let us turn our attention to the description of the tritters alike that of beam splitters in Ref. [24]. The matrix U describing the tritter in Eq. (5.25) can be decomposed into a generalized “Euler-angle” parametrization defined in Ref. [20]:

$$U(\alpha, \beta, \gamma, \theta, a, b, c, \varphi) = e^{(i\hat{\lambda}_3\alpha)} e^{(i\hat{\lambda}_2\beta)} e^{(i\hat{\lambda}_3\gamma)} e^{(i\hat{\lambda}_5\theta)} e^{(i\hat{\lambda}_3 a)} e^{(i\hat{\lambda}_2 b)} e^{(i\hat{\lambda}_3 c)} e^{(i\hat{\lambda}_8\varphi)}. \tag{5.34}$$

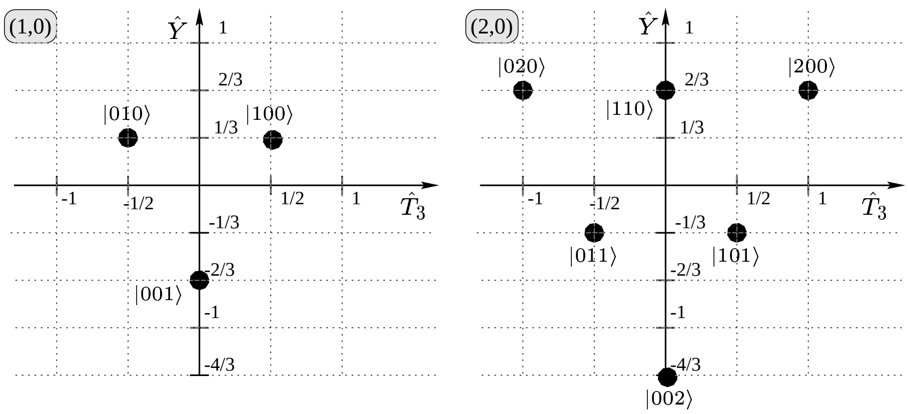

Figure 5.3: $SU(3)$-multiplets $(1,0)$ of one photon and $(2,0)$ of two photons visualized on the $\hat{T}_3 - \hat{Y}$ plane.

This expression enables us to connect a standard parametrization of $SU(3)$ with the physical parameters of the actual realization of the tritter, such as transmittivity and reflectivity coefficients of the optical elements involved.

Similarly to the case of beam splitters, the action of a tritter may be described as follows: the input of the tritter can be described by the operators $\hat{F}_1 \dots \hat{F}_8$, which form a $\mathfrak{su}(3)$ algebra. With these operators, all the others described in Eqs. (5.30) can be expressed and used. The tritter transformation U defined in Eq. (5.25) turns these operators into operators $\hat{G}_1 \dots \hat{G}_8$ describing the output modes. The $\hat{G}$-s are formed from the $\hat{b}_1, \hat{b}_2, \hat{b}_3$ operators. The transformation U is represented by the *adjoint representation* of $SU(3)$, which is an 8-parameter subgroup of the group $SO(8)$ of 8-dimensional rotations:

$$\hat{G}_i = U\hat{F}_i U^\dagger = \sum_{j=1}^{8} R_{ij}\hat{F}_j, \quad R \in SO(8). \tag{5.35}$$

The explicit form of the R matrices is given in Refs. [22, 21].

5.3 Summary

In this chapter I have presented a generalization of a quantum state design scheme based on nonlocality. I have analyzed the teleportation aspect of the arrangement, especially from the point of view of photon number conservation. Inspired by this, I outlined a description of passive lossless linear optical six-ports in terms of $SU(3)$ symmetry.

Chapter 6

Összefoglalás (Resume in Hungarian)

Ebben a fejezetben a *„Kvantumteleportáció általános Hilbert terekben és az optikában”* című doktori disszertációm részletes magyar nyelvű összefoglalása található, az SZTE TTK Doktori Tanácsa által előírt terjedelemben. Ennek megfelelően az összefoglaló nem az angol nyelvű összefoglaló fejezet fordítása. A fejezet pontjai a disszertáció fejezeteit követik.

6.1 Bevezetés

A XX. század vége a kvantummechanika „második aranykorának” tekinthető. Ez az elmélet, amely az 1920-as években született, forradalmat jelentett a fizikában, és fontos mérföldkővé vált az európai gondolkodás általános világképe számára is. Az elmélet rendkívül sikeres volt: használható modellt szolgáltatott az anyag szerkezetét és kölcsönhatásait illetően, magyarázatot adott a fény keletkezésére és anyaggal való kölcsönhatására, és még számos más kérdésre választ adott.

A sikereknek azonban ára volt: elvesztettük a szemléletes, az ember számára elképzelhető fizikát. És noha a kvantummechanika pontos recepteket szolgáltat a mérhető adatok származtatására az absztrakt matematikai modellből, számos paradox, a klasszikus szemlélet számára idegen jelenséggel kell szembesülnünk, melyek mögött rendszerint a kvantummechanika két különös fogalma áll: az *összefonódottság* és a *kvantummechanikai* *mérés*.

Ezen problémák, noha már kezdettől érzékelhetőek voltak (például Einstein, Podolsky és Rosen híres cikke az összefonódásról 1935-ben jelent meg [31]), kezdetben gyakorlati szempontból kevésbé tűntek alapvetőnek. Az utóbbi néhány évtizedben azonban kísérletileg is elérhetővé váltak olyan alapvető kvantummechanikai rendszerek, amelyek az alapjelenségeket tiszta formában mutatják. Az ilyen, elméleti szempontból egyszerűnek tekinthető rendszerek előállítása kísérleti szempontból nehéz, csúcstechnológiát igénylő feladat.

Részben a kísérleti technológia fejlődésének köszönhető a *kvantuminformáció* témakörének kibontakozása, amely a klasszikus információelméletnek egy kvantummechanikai eszközöket alkalmazó általánosítása. Itt az információ alapegysége egy kétállapotú kvantumrendszer, a *kvantumbit* valamely állapota. Míg egy klasszikus bit értéke 0 vagy 1 lehet, a kvantumbit állapota az állapottér két bázisvektorának, vagyis a klasszikus bit állapotainak megfelelő $|0\rangle$ és $|1\rangle$ állapotoknak tetszőleges (normált) lineárkombinációja lehet. A biteknek kvantumbiteket megfeleltetve, és a kvantummechanika szabályait alkalmazva fölépíthető a kvantuminformáció elmélete, amely jelentősen különbözik a klasszikus információelmélettől, sok érdekes problémát és lehetőséget fölvetve. Ugyanakkor a kvantuminformáció, és általában véve az információ fogalmának alkalmazása rendkívül érdekes a fizika számára is. A kvantuminformáció vizsgálatakor rendkívüli módon leegyszerűsített fizikai rendszereket tárgyalunk. Ez azt is lehetővé teszi, hogy a kvantummechanika alapvető aspektusait ezeken a rendszereken tiszta formában vizsgálhassuk.

Disszertációmban a kvantuminformáció manipulálásának egyik alapvető eljárását, a *kvantumteleportációt* elemeztem különféle szempontokból. Ez a fizikai rendszer állapotának megsemmisítése és későbbi előállítása más helyen Einstein-Podolsky-Rosen párok és klasszikus kommunikáció segítségével, melyet eredetileg Charles Bennett javasolt 1993-ban [7]. A kvantumteleportáció az összefonódottság és a kvantummechanikai mérés egyik legérdekesebb alkalmazása.

Megfontolásaim egy része nem kötődik szorosan a jelenség konkrét megvalósításhoz, ezért ezeket „általános Hilbert-terekre” vonatkozó megfontolásoknak tekintem. Másik részük a kvantumteleportáció valamely optikai megvalósításához szorosabban kapcsolódik.

A kutatás célja részben az volt, hogy a teleportáció jelenségét különféle formalizmusokban tárgyaljam, melyek a jelenség más-más részleteire világítanak rá. Emellett megvizsgáltam a kvantumteleportáció egy optikai megvalósítását, és annak alkalmazását kvantumállapot tervezés céljaira.

6.2 A kvantummechanika és kvantumoptika elemei

A disszertáció 1. fejezetének célja, hogy az eredményekhez felhasznált előismereteket összefoglalja, lehetőség szerint önmagukban érthetővé téve a disszertáció további fejezeteit. A kvantummechanika alapjainak összefoglalásakor a legfontosabb szempont az volt, hogy a két- illetve többrészű[1] kvantumrendszerek elméletének elemeit áttekintsem.

Tekintsünk egy kétrészű rendszert, amely az A és B részrendszerekből áll. A kvantummechanika szabályai szerint a kétrészű rendszer állapottere a részrendszerek állapotterének tenzorszorzata, vagyis a rendszer lehet például a

$$\left|\Psi^{(-)}\right\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B\right) \tag{6.1}$$

vektorral leírt állapotban, amely nem írható fel a két részrendszer állapotainak szorzataként. Az ilyen tulajdonságú állapotokat *összefonódottnak* nevezzük. Az összefonódás tipikus kvantumjelenség, amely a köznapi szemlélet számára furcsa szituációkra vezet, de létezése kísérletileg kimutatható. Ilyen paradox jelenség az összefonódott állapotban preparált, de időben és térben eltávolodott részrendszerek viselkedésében jelentkező statisztikus korrelációk jelenléte, amelyek nem magyarázhatók lokális rejtettparaméter-elméletekkel. Ezek felismerése Einstein, Podolsky és Rosen [31], valamint Bell [5] nevéhez fűződik, ezért elnevezésük *EPR korreláció*. Az összefonódottság az egész disszertációban centrális jelentőségel bír.

A 1. fejezet második része az elektromágneses tér kvantálását, a fotodetektálást, a kvázivalószínűségeloszlásokat, a veszteségmentes nyalábosztók kvantumelméletét, és még néhány kvantumoptikai problémakört tekint át röviden, melyek a disszertációban ismertetett optikai eredmények hátterét képezik.

[1] A disszertáció tárgyát képező szakterületnek jelenleg még nincs kialakult, egységesen elfogadott magyar szaknyelve, így bizonyos esetekben kevéssé ismert fordulatokat használok. A „kétrészű” szó az angol „bipartite” szónak egy Diósi Lajostól származó magyarítása, amely két részrendszerből álló rendszert jelent.

6.3 Bevezetés a kvantumteleportációhoz

Tekintsük át a Bennett féle kvantumteleportációs sémát [7] kvalitatíve. A feladó, akit hagyományosan Alice-nak nevez az irodalom, rendelkezik egy fizikai rendszerrel (ez az 1. részrendszer, ld. a 2.1. ábrát), amely egy ismeretlen kvantumállapotban van. Ezt az állapotot kell megsemmisítenünk, és a vevő, Bob oldalán helyreállítanunk. Az állapot megsemmisítése ehhez elengedhetetlen, mivel egy fizikai rendszer „klónozását” tiltják a kvantummechanika törvényei [81]. A teleportáció kivitelezéséhez Alice és Bob előzetesen megosztoznak egy kétrészű rendszer egy-egy részrendszerén (a 2. és és 3. részrendszerek), amelyek összefonódott állapotúak. Alice ezt követően egy olyan mérést végez a nála lévő 1. és 2. részrendszereken, melynek eredményeképp azok egy összefonódott állapotba kerülnek. (A mérés sajátállapotai teljesen összefonódott állapotok.) A mérés hatására a Bobnál lévő 3. részrendszer állapota a teleportálni kívánt állapot valamilyen transzformáltja lesz, ugyanakkor a mérés nem szolgáltat információt a teleportálandó állapotról Alice számára. A transzformációt, amelyet Bobnak el kell végeznie a nála lévő részrendszeren ahhoz, hogy megkapja a teleportálandó állapotot, Alice mérésének eredménye szabja meg. Ezt tehát egy klasszikus csatornán el kell juttatni Bobhoz. A szükséges transzformáció qubitek esetén a Pauli-mátrixokkal írható le.

Bennett a teleportációt kétállapotú rendszerekre, vagyis qubitekre fogalmazta meg, és általánosította azt tetszőleges végesdimenziós Hilbert-térrel leírható rendszerekre. A részletes leírást qubitre a 2.1. részben találjuk.

A folytonos kvantumváltozók, vagyis végtelen dimenziós Hilbert-terek állapotainak teleportálására szolgáló elrendezést Braunstein és Kimble [17, 34] valósította meg elsőként, Vaidman [73] felvetése alapján. A protokoll maga azonos a 2.1. ábrán látható Bennett-féle teleportációéval, de a fizikai rendszerek és az alkalmazott eljárások különböznek. A rendszerek az elektromágneses tér módusai, az összefonódott állapotok pedig a fény kétmódusú összenyomott állapotai. Az unitér transzformáció, melyet Bobnak el kell végezni, fázistérbeli eltolás. Az eltolás mértéke Alice mérési eredményétől függ. A Bennett és a Braunstein-Kimble séma kapcsolatát talán a 3.4. pontban bemutatott Wigner-függvényes leírás mutatja a legvilágosabban.

6.4 Kvantumteleportáció általános Hilbert-tereken

Ebben a részben az általános Hilbert-terekre vonatkozó eredményeket foglalom össze.

6.4.1 A kvantumteleportáció klasszikus határesete

Itt azt vizsgáljuk meg, milyen hatással van a qubit Bennett-féle teleportációjára, ha az alkalmazott összefonódott pár állapota nem egy ideális maximálisan összefonódott állapot. Az állapotok egy olyan családját tekintjük, amely tartalmazza az ideális összefonódott pár állapotát, és annak egyfajta klasszikus határesetét is.

Tegyük fel, hogy az Alice-nál lévő 1. részrendszer teleportálni kívánt állapototát a

$$\rho_{\text{in}}^{(1)} = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \tag{6.2}$$

sűrűségmátrix írja le. Alice és Bob olyan kvantumbiteken osztoznak (2. és 3. részrendszerek), melyek állapota

$$\begin{aligned} \rho^{(23)}(\alpha) = \frac{1}{2}(&\left|\uparrow_2\right\rangle\left|\downarrow_3\right\rangle\left\langle\uparrow_2\right|\left\langle\downarrow_3\right| + \left|\downarrow_2\right\rangle\left|\uparrow_3\right\rangle\left\langle\downarrow_2\right|\left\langle\uparrow_3\right| \\ &-\alpha\left|\uparrow_2\right\rangle\left|\downarrow_3\right\rangle\left\langle\downarrow\right|_2\left\langle\uparrow_3\right| - \alpha\left|\downarrow_2\right\rangle\left|\uparrow_3\right\rangle\left\langle\uparrow_2\right|\left\langle\downarrow_3\right|), \end{aligned} \tag{6.3}$$

ahol $\alpha \in [0,1]$. Itt a bázisállapotokat a spineknél megszokott módon jelöljük. Az $\alpha = 1$ esetben ez a Bennett által alkalmazott maximálisan összefonóditt állapot, tehát az ideális teleportációt kapjuk vissza. Az $\alpha = 0$ esetben a $\left|\downarrow_2\right\rangle\left|\uparrow_3\right\rangle$ és a $\left|\uparrow_2\right\rangle\left|\downarrow_3\right\rangle$ állapotok (inkoherens) *keverékét* kapjuk, amely két klasszikus bit antikorrelált állapotának felel meg. (A Bennett által alkalmazott állapot ugyanakkor e két szorzat-bázisállapot *koherens szuperpozíciója*.) A (6.3) állapotok tehát az inkoherens keveréktől a koherens szuperpozícióig képeznek átmenetet olymódon, hogy az összefonódott állapot sűrűségmátrixának off-diagonális elemeit, vagyis *koherenciáit* α-szorosukra csökkentjük.

A Bennett-féle protokollt változatlanul alkalmazva, de a (6.3) állapotokból kiindulva, a teleportáció végeredménye a

$$\rho^{(3)} = \begin{pmatrix} \rho_{00} & \alpha\rho_{01} \\ \alpha\rho_{10} & \rho_{11} \end{pmatrix} \tag{6.4}$$

állapot lesz. Tehát az összefonódott állapot sűrűségmátrixának off-diagonális elemeinek szorzóját öröklik a teleportált állapot sűrűségmátrixának off-diagonális elemei, koherenciái.

Abban a speciális esetben, ha a bemenő állapot sűrűségmátrixa eleve diagonális:

$$\rho_{\mathrm{in}}^{(1)} = \begin{pmatrix} \rho_{00} & 0 \\ 0 & \rho_{11} \end{pmatrix}, \tag{6.5}$$

akkor az egy klasszikus véletlen bit állapotát írja le, amely ρ_{00} valószínűséggel 0, ρ_{11} valószínűséggel 1 állapotban van. Ezesetben az $\alpha = 0$ „teleportáció” is sikeres. Végiggondolva, ekkor protokollunk épp a klasszikus Vernam-titkosításnak felel meg, ld. a 3.1. ábrát! A Bell-mérés egyszerű kizáró VAGY (XOR) műveletté, az unitér transzformációk bitcserékké egyszerűsödnek. A kvantumteleportáció tehát a Vernam-kód kvantummechanikai általánosításának tekinthető.

A 3.2.3. pontban részletesen elemezünk egy gondolatkísérletet is, amely megmutatja az átmenetet a kvanutmteleportációtól a Vernam-kód felé, szemléltetve a $\alpha \in]0,1[$ köztes eseteket is.

6.4.2 A kvantumteleportáció leírása relatív állapot reprezentációkkal

A disszertáció 3.3. pontja a tiszta állapotokat alkalmazó kvantumteleportáció nagyon tömör, bázisfüggetlen leírását adja antilineáris antiunitér operátorok segítségével, amelyek a kvantumcsatornák (szuperoperátorok) elméletében alkalmazott relatív állapot reprezentációk alapjai.

A leírás kiindulópontja a következő. Tekintsük a $\mathscr{H}_A$ és $\mathscr{H}_B$ Hilbert-tereket! Legyen $|\Phi\rangle_{AB} \in \mathscr{H}_A \otimes \mathscr{H}_B$ kétrészű tiszta állapot. Ez mindenképp kifejthető

$$|\Phi\rangle_{AB} = \sum_{ij} C_{ij} |i\rangle_A \otimes |j\rangle_B \tag{6.6}$$

alakban, ahol a $|i\rangle_A \otimes |j\rangle_B$ vektorok a szorzat ortonormált bázis elemei. Definiálhatunk egy $\mathscr{H}_A \to \mathscr{H}_B$ antilineáris operátort a következőképp:

$$L_{|\Phi\rangle} |i\rangle_A = \sum_j C_{ij} |j\rangle_B. \tag{6.7}$$

Ezzel a kétrészű állapot:

$$|\Phi\rangle_{AB} = \sum_i |i\rangle_A \otimes (L_{|\Phi\rangle} |i\rangle_A). \tag{6.8}$$

Ilymódon minden kétrészű állapotnak megfelel egy antilineáris operátor, és ez a megfeleltetés kölcsönösen egyértelmű. Sőt, belátható, hogy ebben a reprezentációban a (6.8) egyenlet változatlan formában, ugyanazzal az $L_{|\Phi\rangle}$ operátorral érvényes, ha $\mathscr{H}_A$-n egy másik bázist választunk.

Belátható, hogy $|\Phi\rangle_{AB}$ akkor és csak akkor maximálisan összefonódott, ha $L_{|\Phi\rangle}$ *antiunitér*. Ilyen esetben minden $L_{|\Phi\rangle}$-hez tartozik egy úgynevezett relatívállapot-reprezentáció, amely fontos eszköz a kvantumcsatornák elméletében. Ennek részleteit itt nem ismertetjük, mivel nekünk most maga az $L_{|\Phi\rangle}$ operátor érdekes.

Kiemeljük azt, hogy a kétrészű állapotok részleges nyoma is könnyen kifejezhető ebben a reprezentációban:

$$\begin{aligned} L^{\star}_{|\Phi\rangle_{AB}} L_{|\Phi\rangle_{AB}} &= \mathrm{Tr}_B \,|\Phi_{AB}\rangle\,\langle\Phi_{AB}| = \rho_A \\ L_{|\Phi\rangle_{AB}} L^{\star}_{|\Phi\rangle_{AB}} &= \mathrm{Tr}_A \,|\Phi_{AB}\rangle\,\langle\Phi_{AB}| = \rho_B. \end{aligned} \tag{6.9}$$

Ezek után rátérhetünk a teleportáció leírására. Tegyük fel, hogy az Alice által végzett kollektív mérés sajátállapotai a

$$|\sigma_q\rangle = \sum_i (L_q|i\rangle) \otimes |i\rangle, \tag{6.10}$$

kétrészű állapotok, és a $L_{|\sigma\rangle}$ operátorral megadott állapoton osztoznak. Feltéve, hogy Alice mérése a q eredményt adja, a Bobnál lévő állapot a mérés után így írható:

$$\rho_{\mathrm{out}} = \frac{LL^{\star}_q \rho_{\mathrm{in}} L_q L^{\star}}{\mathrm{tr}_A(L_q L^{\star} L L^{\star}_q \rho_{\mathrm{in}})}. \tag{6.11}$$

ahol a $\star$ az antiunitér konjugáltat jelenti. *Ez a teleportáció tömör, bázisfüggetlen leírása.* Az adott mérési kimenetel esetén a teleportáció akkor hajtható végre, ha ebből valamilyen (lehetőleg lineáris, unitér) transzformációval megkapható a teleportálni kívánt állapot. Kifejezhető a q mérési eredmény valószínűsége is:

$$p_q(\rho_{\mathrm{in}}) = \mathrm{tr}_A \left(L_q L^{\star} L L^{\star}_q \rho_{\mathrm{in}} \right) \tag{6.12}$$

. Könnyen megmutathatjuk azt is, hogy a (6.11) szuperoperátor invertálható, vagyis a q mérési eredmény esetén a teleportáció sikerrel végrehajtható, akkor a (6.11) akkor és csak akkor lineáris, ha a (6.12) valószínűség független a ρ_{in} állapottól.

Ez a formalizmus alapot szolgáltatott a nem teljesen összefonódott állapotok feltételes teleportációjának tárgyalásához [50].

6.4.3 Kvantumteleportáció diszkrét Wigner-függvényekkel

Minden kvantummechanikai jelenség leírható a fázistéren értelmezett kvázivalószínűségekkel: ez a sűrűségmátrixokkal való direkt leírás alternatívája. Mivel a kanonikusan konjugált mennyiségek nem mérhetőek egyszerre, a fázistér használata valamivel összetettebb feladat, mint klasszikus esetben: az eloszlásfüggvény definíciója nem egyértelmű, és az állapotot leíró eloszlásfüggvények bizonyos esetekben negatív értékeket is felvehetnek. Ezért nevezzük az ilyen függvényeket *kvázivalószínűségeloszlásnak*. A kvázivalószínűségeloszlásfüggvények közül gyakran használjuk az úgynevezett Wigner-függvényeket, amelyek bizonyos értelemben a klasszikus eloszlásfüggvényhez hasonlóan viselkednek. A végtelendimenziós Hilbert-térrel leírható rendszerek (például a harmonikus oszcillátor) esetén a Wigner-függvények, melyeket a 1.3.5. részben ismertettem, a kvantumoptika szokásos eszköztárába sorolhatók.

Végesdimenziós Hilbert-terek esetén a fázistér és az eloszlásfüggvények bevezetése több problémát fölvet, így azok használata kevésbé elterjedt. Jóllehet, mint látni fogjuk, sikerrel alkalmazhatóak például a kvantumteleportáció tárgyalására. Ehhez először vezessük be a szükséges matematikai apparátust.

A fázistér bevezetésének egyik nehézsége abban áll, hogy végesdimenziós Hilbert-terekben nem lehet a kanonikus kvantálás szabályának megfelelően kanonikusan konjugált mennyiségeket bevezetni: a Heisenberg-féle kommutációs relációk csak triviálisan elégítetők ki. (Ugyanez volna a helyzet végtelendimenziós eseben is, ha az egész téren értelmezett operátorokkal dolgoznánk. A kvantummechanika operátorai a Hilbert-tér sűrű alterén értelmezettek.

A végesdimenziós fázistéren értelmezett „hely” és impulzus operátorokat bevezethetjük a sajátállapotok definiálásával: egy N dimenziós Hilbert-téren legyenek

$$\hat{q} = \sum_{k=0}^{N-1} k|k\rangle\langle k|, \qquad \hat{p} = \sum_{l=0}^{N-1} l|p_l\rangle\langle p_l|, \tag{6.13}$$

és követeljük meg, hogy

$$|p_l\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{i\frac{2\pi}{N}kl}|k\rangle. \tag{6.14}$$

Ez analóg az impulzusállapot szokásos helyreprezentációbeli alakjával. Az előbbi formulában, és ebben a részben végig az egész számok aritmetikai műveletei modulo N értendők.

A végesdimenziós Wigner-függvényt Wootters vezette be elsőként [80], mi az általa javasolt formalizmust használjuk. Ez különösen egyszerű, ha N prímszám. Ha N összetett szám, a fázistér ebben a formalizmusban Descartes szorzat alakú, jó példa lesz erre a teleportációban szereplő háromrészű rendszer. A továbbiakban feltesszük, hogy $N \geq 3$ prímszám. Ha $N = 2$, az operátorok más alakúak. Definiáljuk a

$$\hat{A}(q,p) = \sum_{r,s} \delta_{2q,r+s} e^{i\frac{2\pi}{N}p(r-s)} |r\rangle\langle s|, \tag{6.15}$$

diszkrét Wigner-operátorokat. Ezekkel egy ρ sűrűségmátrixú állapot Wigner-függvénye:

$$W(q,p) = \frac{1}{N}\,\mathrm{tr}(\rho\hat{A}). \tag{6.16}$$

A q, p számok egy $N \times N$-es rácsot feszítenek ki, ez maga a fázistér.

Megmutatható, hogy az így definiált Wigner-függvény a szokásos Wigner-függvényekéhez hasonló tulajdonságokkal rendelkezik, például *marginálisai* a $\hat{q}$ és $\hat{p}$ mennyiségek mérési statisztikáit szolgáltatják:

$$P_q(q) = \sum_p W(q,p), \qquad P_p(p) = \sum_q W(q,p). \tag{6.17}$$

Többrészű rendszerek esetén a Wigner-operátorok tenzorszorzataival definiáljuk a Wigner-függvényt, például kétrészű rendszerre:

$$W(q_1,p_1,q_2,p_2) = \frac{1}{N^2}\,\mathrm{Tr}(\rho^{(12)}\hat{A}_1(q_1,p_1) \otimes \hat{A}_2(q_2,p_2)). \tag{6.18}$$

A részleges nyom képzése a részrendszer állapotának meghatározása céljából az elhagyandó részrendszer változóiban való átlagolással történik:

$$\begin{aligned} W(q_1,p_1) &= \sum_{q_2,p_2=0}^{N-1} W(q_1,p_1,q_2,p_2), \\ W(q_2,p_2) &= \sum_{q_1,p_1=0}^{N-1} W(q_1,p_1,q_2,p_2). \end{aligned} \tag{6.19}$$

Szükségünk lesz még a két N dimenziós Hilbert-térrel leírható kétrészű rendszer Bell-állapotaira, ezek maximálisan összefonódott állapotok, amelyek teljes bázist alkotnak a $\mathscr{H} \otimes \mathscr{H}$ szorzattéren:

$$|\Xi_{P,X}\rangle_{12} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{i\frac{2\pi}{N}kP} |k\rangle_1 |k-X\rangle_2. \tag{6.20}$$

Ezek az állapotok a $\hat{X} = \hat{x}_1 - \hat{x}_2$ *relatív koordináta*, és a $\hat{P} = \hat{p}_1 + \hat{p}_2$ *teljes impulzus* közös sajátállapotai: ezek ugyanis kompatibilis (egyszerre mérhető) mennyiségek.

Ezzel készen állunk a Bennett-féle teleportációs séma fázistérbeli leírására, ahol a részrendszerek $N \geq 3$ prím dimenziójú Hilbert-terekkel írhatók le.

Tegyük fel, hogy a teleportálandó állapot Wigner-függvénye $W_{\text{in}}(q_1, p_1)$ (1. részrendszer). A Bennett által használt maximálisan összefonódott állapot, melyet Alice és Bob megoszt (2. és 3. részrendszerek), a $|\Xi_{0,0}\rangle_{23}$ állapot, melynek Wigner-függvénye:

$$W_{\text{EPR}}(q_2, p_2, q_3, p_3) = \frac{1}{N^2} \delta_{q_2, q_3} \delta_{p_2, -p_3}. \tag{6.21}$$

(Az EPR index itt Einstein, Podolsky és Rosen nevére utal.) Ez nagyon szemléletes eredmény: az eloszlás éppen a nulla teljes impulzusú és relatív koordinátájú állapot eloszlásának felel meg. A teljes rendszer állapota tehát a

$$W(q_1, p_1, q_2, p_2, q_3, p_3) = \frac{1}{N^2} W_{\text{in}}(q_1, p_1) \delta_{q_2, q_3} \delta_{p_2, -p_3} \tag{6.22}$$

Wigner-függvénnyel írható le. A következő lépésben Alice megméri a nála lévő részrendszerek (1. és 2.) relatív koordinátáját és teljes impulzusát. Ennek leírására egy kanonikus transzformációval olyan változókra térünk át, amelyek közt e két mennyiség szerepel, és a konjugált mennyiségekben átlagolunk. Kiszámítható, hogy

- A mérés után a Bobnál lévő rendszer állapota

$$W_{\text{out}}(q_3, p_3) = W_{\text{in}}(q_3 + X_2, p_3 + P_1), \tag{6.23}$$

  ahol X_2 és P_1 az Alice által végzett mérés eredményei.

- Az Alice által végzett mérés statisztikája egyenletes eloszlást mutat: Alice nem jut információhoz a teleportálni kívánt rendszer állapotáról.

Láthatjuk tehát, hogy a Bob által végrehajtandó inverz transzformáció egy *fázistérbeli eltolás*, ami megfelel a folytonos változók esetén ismert eredménynek. (Ne feledkezzünk meg a modulo N aritmetikáról. A diszkrét eltolás szemléltetésére ld. a 3.4. ábrát.)

Ha a részrendszerek dimenziószáma nem $N \geq 3$ prím, a megfontolás kevésbé látványos, de fizikailag nem ad más eredményt.

Mindebből azt a következtetést vonhatjuk le, hogy a végesdimenziós Wigner-függvény formalizmus kiválóan alkalmas a kvantumteleportáció leírására, így hasznos lehet a kvantuminformáció területén is. Másrészt a megfontolás világos kapcsolatot teremt a Bennett és a Braunstein-Kimble séma között, noha a véges-végtelen határátmenet egzakt matematikai elvégzése nem triviális.

6.5 A fénymódus teleportációjának leírása koherens állapotokkal

A disszertáció 4. fejezete a folytonos kvantumváltozók teleportációjának Braunstein és Kimble által bevezetett módszerének egy alternatív leírásával foglalkozik, melynek kiindulópontja az alacsonydimenziós koherens állapot reprezentációk alkalmazása. A teleportációban résztvevő fizikai rendszerek itt az elektromágneses tér módusai (harmonikus oszcillátorok), melyek állapotait tárgyalásomban koherens állapotok szuperpozícióiként írom le.

Elsőként a

$$\hat{q} = \frac{\hat{a}+\hat{a}^\dagger}{2}, \quad \hat{p} = \frac{\hat{a}-\hat{a}^\dagger}{2i}. \tag{6.24}$$

kvadratúra-operátorok sajátállapotait írjuk fel. Ezek felfoghatók a megfelelő kvadratúrában végtelen összenyomottságú kvadratúra összenyomott állapotként. A kvadratúra összenyomott állapot a [45] cikk alapján felírható a fázistér megfelelő egyenesei mentén elhelyezett koherens állapotok szuperpozícióiként, a következő módon:

$$\begin{aligned} |\text{Sq. vac. p}\rangle &= \mathcal{N}(r)\int_{-\infty}^{\infty} dx G_r(x)\,|x\rangle\,, \\ |\text{Sq. vac. x}\rangle &= \mathcal{N}(r)\int_{-\infty}^{\infty} dy G_r(y)\,|iy\rangle\,, \end{aligned} \tag{6.25}$$

ahol

$$\mathcal{N}(r) = \frac{1}{\sqrt{\pi}}\frac{e^{r/2}}{\sqrt{e^{2r}-1}}, \quad \text{and} \quad G_r(x) = e^{-\frac{|x|^2}{e^{2r}-1}} \tag{6.26}$$

Gauss súlyfüggvény. Az r itt az összenyomottságra jellemző paraméter. Az $r \to \infty$ határesetben a kvadratúrák nulla sajátértékeihez tartozó állapotait kapjuk:

$$||P=0\rangle = \lim_{r\to\infty}\int_{-\infty}^{\infty} dx G_r(x)\,|x\rangle = \int_{-\infty}^{\infty} dx\,|x\rangle$$

$$||X=0\rangle = \lim_{r\to\infty}\int_{-\infty}^{\infty} dy G_r(y)\,|iy\rangle = \int_{-\infty}^{\infty} dy\,|iy\rangle\,, \tag{6.27}$$

amelyek nem normálhatók. Végül a kvadratúra-sajátállapotok az előbbi állapot fázistérbeli eltolásával kapjuk, melyet a $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ Glauber-féle eltolási operátorral valósíthatunk meg:

$$||P\rangle = \hat{D}(iP)||P=0\rangle = \int_{-\infty}^{\infty} dx e^{ixP}\,|x+iP\rangle$$

$$||X\rangle = \hat{D}(X)||X=0\rangle = \int_{-\infty}^{\infty} dy e^{-iXy}\,|X+iy\rangle\,. \tag{6.28}$$

Rendelkezésre állnak tehát az elektromágneses tér két kanonikusan konjugált mennyiségéhez tartozó állapotok.

Tekintsünk most két módust. Az ezek állapotterét kifeszítő teljesen összefonódott Bell-bázis, amelyet a teleportációnál használhatunk, a

$$\left|\Psi_{\mathrm{prod}_{X,P}}\right\rangle = ||X\rangle_1||P\rangle_2. \tag{6.29}$$

szorzatbázis transzformációjával kaphatók. Ez a transzformáció egy szimmetrikus nyalábosztó hatásának felel meg, ami összevág a Bell-állapot detektor kísérleti megvalósításával: az Alice-nél lévő módusok egy nyalábosztón interferálnak, és az így keletkező két módus X és P kvadratúráját mérik homodin detektorral. (A detektálást itt ideális projektív mérésnek tekintjük.)

A számítást elvégezve, a Bell-bázis elemei a

$$\gamma := \frac{x+iy}{\sqrt{2}}, \quad A := \frac{X+iP}{\sqrt{2}} \tag{6.30}$$

komplex változók segítségével az alábbi alakúak:

$$|B(X,P)\rangle = \int d^2\gamma e^{A\gamma^* - A^*\gamma}|\gamma + A\rangle_1|\gamma^* - A^*\rangle_2\,. \tag{6.31}$$

Az A komplex szám indexeli az állapotokat. Vegyük észre, hogy noha két módus állapotáról van szó, egyetlen komplex számsíkra integrálunk.

A teleportáció menete a következő: Alice és Bob az $A = 0$ összefonódott állapoton osztoznak, ez az összefonódott pár (2. és 3. részrendszerek). Ez az előbbiek szerint

$$|\Psi_{\mathrm{EPR}}\rangle_{23} = \int d^2\alpha G_r(\sqrt{2}|\alpha|)|\alpha^*\rangle_2|\alpha\rangle_3. \tag{6.32}$$

Az Alice-nál lévő teleportálandó állapotot (1. részrendszer) állapotát a Glauber-féle analitikus reprezentációban írjuk fel:

$$\left|\Psi_{\mathrm{in}}\right\rangle_1 = \int d^2\beta e^{-\frac{|\beta|^2}{2}} f(\beta^*)\left|\beta\right\rangle_1. \tag{6.33}$$

Mint a disszertációban megmutatom, ebből a kiindulásból a teleportáció folyamata kényelmesen nyomon követhető. A lépések: Alice projektív mérése, mely az A paraméterű állapotra vetít, és a Bob által végrehajtott unitér transzformáció, amely az Alice által kapott A mért értéktől függ. A számítás kulcslépése a

$$\frac{1}{\pi}\int d^2\alpha e^{-|\alpha|^2+\alpha\beta^*} f(\alpha^*) = f(\beta^*) \tag{6.34}$$

Glauber-integrál alkalmazása.

Abban az ideális esetben, ha végtelenül összenyomott állapotokból indulunk ki, a Bobnál lévő állapot Alice mérése után a teleportálandó állapot eltoltja:

$$|\Psi_{\mathrm{f}}\rangle = D(-2A)|\Psi_{\mathrm{in}}\rangle, \tag{6.35}$$

összhangban Braunstein és Kimble eredményeivel. Ha az összefonódott pár nem teljesen összenyomott állapotokból alakul ki, vagyis nem teljesen összefonódott, akkor a teleportáció végeredménye (a Bob által végrehajtott transzformáció után):

$$|\Psi_{\mathrm{f}}\rangle = \int d^2\alpha G_r(\sqrt{2}|\alpha-2A|)e^{\frac{-|\alpha|^2}{2}} f(\alpha^*)|\alpha\rangle. \tag{6.36}$$

Látható, hogy egy Gauss-típusú simító faktor jelenik meg, ami nem csak az összenyomottságot (és így az összefonottságot is) leíró r paramétertől, hanem az A mérési eredménytől is függ. Ez annak a következménye, hogy az összefonódott állapot véges számú fotont tartalmaz.

Összefoglalva: megmutattam, hogy a folytonos változók Braunstein-Kimble féle kvantumteleportációja leírható direkt módon, koherens állapotok speciális szuperpozícióinak alkalmazásával.

6.6 Állapottervezés teleportációval: fotoninterferencia

A nemlineáris optikai folyamatokban keletkező fény rendszerint kis számú fotont tartalmaz. Ezen fotonok manipulálhatók lineáris optikai eszközökkel (unitér műveletek), és fotodetektorok segítségével kvantummechanikai mérés végezhető rajtuk. Az elektromágneses tér módusai, mint részrendszerek, alkalmasak a kvantuminformáció fizikai reprezentálására. Ezeket az elrendezéseket „kevésfotonos interferencia eszközöknek” fogom nevezni. A kvantumteleportáció első kísérleti megvalósítása is ilyen módon történt [15], de az alkalmazások közé sorolhatjuk a kvantumlitográfiát [14] is.

Disszertációm 5. fejezete kevésfotonos interferencia eszközökkel foglalkozik: egy teleportációra épülő kvantumállapot tervező séma általánosításával, és optikai hatportokkal.

6.6.1 Az általánosított kvantumolló

A „kvantumollót” Pegg, Phillips és Barnett javasolta [62], az elektromágneses tér egy haladóhullámú módusának kvantumállapot-manipulációjára. Az elrendezés két szimmetrikus nyalánosztóból, és két fotodetektorból áll, és arra alkalmas, hogy a bejövő módus állapotából „levágja” az egynél több fotonból álló komponenseket, a vákuum és az egyfotonos Fock-állapot szuperpozícióját hozva létre. Az eszköz működése alapvetően a kvantumteleportációra épül.

Munkámban a kvantumolló olyan általánosítását vizsgáltam, amely a kétfoton-állapotig, illetve az N-foton állapotig képes levágni. Ez egyben az N dimenziós Hilbert-téren való teleportáció egy lehetséges optikai megvalósítása.

Az elrendezés az 5.1. ábrán látható. A továbbiakban az ábrán szereplő jelöléseket használom a módusok, illetve azok eltűntető operátorainak jelölésére. A bemenetre érkező állapot fotonszám-reprezentációban

$$|\Psi_{\text{in}}\rangle = \sum_{k=0}^{\infty} \gamma_k |k\rangle. \tag{6.37}$$

A BS_1 nyalábosztó a bemenetére érkező $|\Psi_{12}\rangle$ állapotból egy összefonódott állapotot állít elő. Feltételezzük $|\Psi_{12}\rangle$ állapot a $|11\rangle$ kétmódusú Fock-állapot, amely parametrikus konverzióval előállítható. A BS_1 nyalábosztó kimenetein egy

$$|\Psi'_{12}\rangle = \beta_0|20\rangle + \beta_1|11\rangle + \beta_2|02\rangle \tag{6.38}$$

állapot jelenik meg, melynek β paraméterei a BS_1 paramétereitől (átlátszóság, a transzmittált és a reflektált nyaláb fázistolása) függenek. E két módus egyike a kimenő állapot, a másik módus pedig a BS_2 nyalábosztón a bemenő állapottal interferál. Teleportáció szempontjából tekintve a $|\Psi'_{12}\rangle$ állapot az összefonódott párnak felel meg, a kimeneten keletkezik a teleportálandó állapot, a BS_2 nyalábosztó pedig a Bell-állapot detektor része.

A BS_2 nyalábosztó kimeneteit egy-egy fotodetektor figyeli. Feltételezzük, hogy ezek ideális fotonszámlálók, amelyek koincidenciában való megszólalásuk esetén a $|11\rangle$ Fock-állapotra való projekciót valósítanak meg a BS_2 kimenő módusain. (Ez egy igen erős feltevés: a jelenleg kísérletileg is elérhető detektorok nem ideális fotonszámlálók.) A berendezés működését sikeresnek tekintjük, ha ez a detektálási esemény következik be.

Megmutattam, hogy amennyiben

- a két nyalábosztó fázistolásai megegyeznek,
- és mindkét nyalábosztó átlátszósága 0.21 vagy 0.79,

akkor a kimeneten sikeres működés esetén a

$$|\Psi_{\mathrm{out}}\rangle \propto \sum_{k=0}^{2} \gamma_k |k\rangle \tag{6.39}$$

„levágott” állapot jelenik meg. A sikeres működés maximális valószínűsége $1/9$, ami akkor érhető el, ha $|\Psi_{\mathrm{in}}\rangle = |\Psi_{\mathrm{out}}\rangle$, ami a (6.39) egyenletbeli állapot feltételes teleportációjának felel meg. (Feltételes teleportáció alatt azt értjük, hogy csak egy bizonyos detektálási esemény, esetünkben az 1-1 foton egyidejű detektálása esetén tekintjük a műveletet sikeresnek.) A többi eseben a Fock-kifejtésnek csak az „elejét” teleportáljuk.

Megmutattam, hogy lehet az eszközt általánosítani tetszőleges N-foton állapotig történő levágásra is, ezt a disszertáció 5.1.3. pontja tartalmazza.

Térjünk vissza a két fotonig való levágás esetére, és vizsgáljuk részletesen a $|\Psi_{\mathrm{in}} = |\Psi_{\mathrm{out}}$ esetet, vagyis amikor az eszköz feltételes kvantumteleportációt valósít meg. A bemenő állapot tartalmazzon tehát maximum két fotont:

$$|\Psi_{\mathrm{in}}\rangle = A_0|0\rangle_3 + A_1|1\rangle_3 + A_2|2\rangle_3, \tag{6.40}$$

Feltehető a kérdés, hogy mely detektálási események esetén lehet teleportációról beszélni, vagyis mikor tartalmaz a kimenet mérés utáni állapota a bejövő állapottal azonos információt. Ennek megválaszolását megkönnyíti az impulzusmomentum Schwinger-reprezentációjának alkalmazása, melyről a nyalábosztókkal kapcsolatban e disszertáció 1.3.8. pontjában olvashatunk.

A lényeges itt számunkra az, hogy a kétmódusú Fock-állapotok (és így az egyes mérési kimenetelek, melyek az ezekre való vetítést jelzik), $SU(2)$ multiplettekbe rendezhetők, amelyek *a nyalábosztó-transzformáció invariáns alterei*. A Casimir-operátor itt az összfotonszámmal fejezhető ki, így az egy multipletthez tartozó állapotok összfotonszáma azonos; a multipletteket az összfotonszám felével indexeljük. A multipletten belüli indexelésre a fotonszám különbsége ad módot. (Az impulzusmomentum terminológiájával a fotonszám fele a teljes spin, míg a különbség fele a spin z-komponens kvantumszámának felel meg. Bevezetve a

$$ {}^{2l}\hat{M}_{l_3} = \hat{c}_2^{\dagger l+l_3}\hat{c}_3^{\dagger l-l_3}, \quad l_3 = -l \dots l \tag{6.41} $$

operátorokat, a komplex együtthatók egy ${}^{2l}\mathscr{A}_{l_3}$, $l_3 = -l \dots l$ készletéhez definiálhatjuk a

$$ {}^{2l}\hat{\mathscr{M}}_{\mathscr{A}} = \sum_{l_3=-l}^{l} {}^{2l}\mathscr{A}_{l_3}\, {}^{2l}\hat{M}_{l_3} \tag{6.42} $$

operátorokat.

A detektorok működése (projektív mérés) előtt a három módus állapota

$$ |\Psi_{\mathrm{m}}\rangle_{123} = \hat{A}^\dagger(\hat{c}_2^\dagger, \hat{c}_3^\dagger, \hat{b}_1^\dagger)|0\rangle, \tag{6.43} $$

alakba írható, ahol a $\hat{A}^\dagger(\hat{c}_2^\dagger, \hat{c}_3^\dagger, \hat{b}_1^\dagger)$ a keltőoperátorok polinomja, amely a $|\Psi_m\rangle_{123}$ állapotot kelti a vákuumból. Ez az operátor kifejezhető a (6.42) egyenletben bevezetett operátorokkal. Miután maximum négy foton lehet egyszerre a kimeneten, az operátor alakja

$$ \begin{aligned} \hat{A}^\dagger(\hat{c}_2^\dagger, \hat{c}_3^\dagger, \hat{b}_1^\dagger) = \quad & A_0({}^2\hat{\mathscr{M}}_{\mathscr{A}} + {}^1\hat{\mathscr{M}}_{\mathscr{B}}\hat{b}_1^\dagger + {}^0\hat{\mathscr{M}}_{\mathscr{C}}\hat{b}_1^{\dagger 2}) \\ + \quad & A_1({}^3\hat{\mathscr{M}}_{\mathscr{D}} + {}^2\hat{\mathscr{M}}_{\mathscr{E}}\hat{b}_1^\dagger + {}^1\hat{\mathscr{M}}_{\mathscr{F}}\hat{b}_1^{\dagger 2}) \\ + \quad & \frac{A_2}{\sqrt{2}}({}^4\hat{\mathscr{M}}_{\mathscr{G}} + {}^3\hat{\mathscr{M}}_{\mathscr{H}}\hat{b}_1^\dagger + {}^2\hat{\mathscr{M}}_{\mathscr{I}}\hat{b}_1^{\dagger 2}). \end{aligned} \tag{6.44} $$

Az írott betűvel jelzett együtthatók a nyalábosztók paramétereitől függnek. Hogy az állapot rekonstruálható legyen, olyan mérési eseményt kell választanunk, amely az A_0, A_1 és A_2 együtthatójú multiplettekben is jelen van. Ezek a ${}^2\hat{\mathscr{M}}_{\mathscr{A}}, {}^2\hat{\mathscr{M}}_{\mathscr{E}}, {}^2\hat{\mathscr{M}}_{\mathscr{I}}$ multiplettek. Vagyis az állapot átviteléhez két fotont kell detektálnunk.

Ez az analízis a fotonszám-megmaradás egy elegáns megfogalmazása segítségével megvilágítja az általánosított kvantumolló teleportáló berendezésként való működését

6.6.2 A passzív lineáris optikai hatportokról

Az előző pontban láttuk, hogy az impulzusmomentum Schwinger reprezentációja, és a nyalábosztók erre alapuló leírása hasznos lehet egy interferometrikus berendezés működésének megértésekor. Felmerül a kérdés, lehetséges-e egy passziv veszteségmentes lineáris optikai hatport (három bemenetű, három kimenetű eszköz, „*tritter*” hasonló leírása.

Az általános tritter $SU(3)$ transzformációt valósít meg a kimenetek és bemenetek eltűntető operátorai között:

$$\hat{b}_i = \sum_{j=1}^{3} U_{ij}\hat{a}_j, \quad i = 1,2,3 \quad U \in SU(3). \tag{6.45}$$

A nyalábosztóhoz hasonló tárgyaláshoz az $\mathfrak{su}(3)$ algebra bozon-reprezentációját kell fölírnunk. Ehhez a $\hat{\lambda}_i$ $i = 1\dots8$ Gell-Mann mátrixokból indulunk ki, melyek az algebra legalacsonyabb dimenziós hű reprezentációját adják, explicit alakjuk a (5.26) egyenletben látható. A bemenet operátoraiból képezhető $\mathfrak{su}(3)$ reprezentáció:

$$\hat{F}_i = \frac{1}{2} \begin{pmatrix} \hat{a}_1^\dagger & \hat{a}_2^\dagger & \hat{a}_3^\dagger \end{pmatrix} \begin{pmatrix} & & \\ & \hat{\lambda}_i & \\ & & \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{pmatrix}, \tag{6.46}$$

a kimenet operátoraiból hasonló $\hat{G}_i$ operátorokat képezhetünk. Ebből kiindulva az $\mathfrak{su}(3)$ minden operátora (léptetőoperátorok, stb.) megkonstruálhatóak, ld. a (5.30) egyenletet.

Felmerül a kérdés, hogy mik alkotják az $SU(3)$ multipletteket ebben az esetben. Az $SU(3)$ csoportnak két Casimir-operátora van, sejthetjük, hogy az egyik a fotonszámmal kapcsolatos, kérdés ugyanakkor, mit ír le a másik.

Az $SU(3)$ multipletteket a T_3–Y síkon szokás ábrázolni, ahol a mi reprezentációnkban

$$\hat{T}_3|nlm\rangle = \frac{1}{2}(n-l)|nlm\rangle, \quad \hat{Y}|nlm\rangle = \frac{1}{3}(n+l-2m)|nlm\rangle. \tag{6.47}$$

Az irodalomból kiderül (ld. pl. [53]), hogy a bozonok nem realizálnak minden multiplettet, ami a bozonok felcserélési szimmetriájának köszönhető. Ebből következően bozonokkal csak a T_3–Y síkon háromszöggel ábrázolható multiplettek realizálhatók bozonokkal. Tipikus multipletteket látunk a 5.3. ábrán, míg a 5.2 ábra a léptető operátorok hatását mutatja a T_3–Y síkon. Utóbbin megfigyelhető, hogy az $\mathfrak{su}(3)$ három $\mathfrak{su}(2)$ részalgebrát tartalmaz. A konklúzió az, hogy bozonok esetén az $SU(3)$ multiplettek szintén egy paraméterrel indexelhetők, ez pedig a teljes fotonszám.

A multiplettek leírása után rátérhetünk a tritter leírására. A (6.45) egyenlet unitér operátora az eltűntető operátorokat transzformálja. Az ezekből képzett $\hat{F}_i \in \mathfrak{su}(3)$ operátorokat az $SU(3)$ adjungált reprezentációjának megfelelő R eleme viszi át a kimenet $\hat{G}_i \in \mathfrak{su}(3)$ operátoraiba:

$$\hat{G}_i = U\hat{F}_i U^\dagger = \sum_{j=1}^{8} R_{ij}\hat{F}_j, \quad R \in SO(8). \tag{6.48}$$

Az adjungált reprezentáció a valós nyolcdimenziós forgatások egy nyolcparaméterű részcsoportját alkotja. (V. ö. nyalábosztó esetén ezek háromdimenziós valós forgatások.) Az R mátrixok explicit alakja megtalálható a [22, 21] cikkekben. Azt mondhatjuk tehát, hogy a trittereknek lehetséges egy a nyalábosztókéhoz hasonló csoportelméleti tárgyalása, noha ez kevésbé praktikus mint a nyalábosztóké: itt nem három, hanem nyolcdimenziós forgatásokkal kell dolgozni.

6.7 Az eredmények tézisszerű összefoglalása

Disszertációmban a kvantumteleportációval és a fotoninterferenciával kapcsolatos eredményeimet mutattam be. Munkám a kvantumkommunikáció elméletének témaköréhez, és a kvantumoptikához kapcsolódik.

A kvantumteleportáció a kvantumkommunikáció alapvető alkotóeleme, és az Einstein-Podolsky-Rosen korrelációk egyik legérdekesebb alkalmazása. Bemutattam ennek egy klasszikus határesetét, amely elősegíti a kvantummechanikai jelenség megértését. Alternatív leírásokat dolgoztam ki a kvantumteleportáció folyamatára. Mindegyik különböző leírás egy-egy sajátos nézőpontot jelent, ezek a nézőpontok a teleportációs folyamat különféle aspektusait fedik fel.

Optikai elrendezéseket tanulmányoztam, amelyekben kis fotonszámú kvantumállapottal rendelkező fénymódusok interferálnak. Ezeken belül egy kvantumállapot-preparáló elrendezés általánosításával foglalkoztam, amelynek alapja a kvantumteleportáció egy optikai megvalósítása. Azt találtam, hogy a veszteségmentes nyalábosztók ismert, az $SU(2)$ szimmetriára épülő leírása segít a szóban forgó interferometrikus elrendezés működésének megértésében. Ez motiválta a passzív, lineáris, veszteségmentes optikai hat-portok (tritterek) $SU(3)$ szimmetria segítségével való tárgyalását.

A főbb új tudományos eredményeimet az alábbiakban pontokba szedve foglalom össze. A kapcsolódó publikációimat a „List of related publications” című oldal tartalmazza.

1. Megmutattam, hogy a kvantumbit Bennett-féle teleportációja a Vernam-titkosítás kvantummechanikai általánosítása. Ez utóbbi az elméletileg legmegbízhatóbb klasszikus titkosítási eljárás, amely azonban végtelen sok korrelált valódi véletlen bitpárt igényel, így a gyakorlatban kevéssé használható.

   A Vernam-titkosítás és a kvantumteleportáció viszonyának megértése hozzásegít a kvantummechanikai folyamat természetének megértéséhez. A klasszikus-kvantum átmenetet egy gondolatkísérlet elemzésén keresztül mutattam be. [II]

2. Vizsgáltam a tiszta állapotú összefonódott párt és projektív mérést alkalmazó, probabilisztikus teleportációs sémákat végesdimenziós Hilbert-tereken. Létezik a kétrészű (két részrenszerből álló) kvantumrendszerek állapotainak egy antiunitér ope-

rátorokkal történő reprezentációja. Ebben a formalizmusban a maximálisan összefonódott állapotokat antiunitér operátorok írják le. Az antiunitér operátorokat a fizikában az időtükrözés leírására szokás használni. Ez egy másik érdekes alkalmazásuk, amely több szempontból előnyös, például teljesen bázisfüggetlen. Ez a reprezentáció a kvantumállapotok és kvantumcsatornák relatívállapot-reprezentációjának is alapjául szolgál. Az antilineáris reprezentáció felhasználásával kifejeztem a szóban forgó teleportáció által megvalósított kvantumcsatornát operátorként, egy tömör, bázisfüggetlen alakban. [VI]

3. A kvantumjelenségek alternatív leírásának elterjedt módja a kvázivalószínűségeloszlás-függvények használata. Az egyik leggyakrabban használt kvázivalószínűségeloszlás a Wigner-függvény. A folytonos változók (oszcillátor állapotok) kvantumteleportációját is Wigner-függvény formalizmusban vezették be először, míg Bennett eredeti cikkében közvetlenül egy végesdimenziós állapottér vektorait használja.

   A végesdimenziós Hilbert-tereken többféleképp is definiálhatunk Wigner-függvényeket. Egy definíciót ezek közül kiválasztva, Bennett-féle kvantumteleportáció leírását adtam végesdimenziós Wigner-függvény formalizmusban. Az új leírás világos képet ad a kvantumteleportáció folyamatáról a végesdimenziós fázistéren, megmutatva a Bennett féle teleportáció kapcsolatát a folytonos kvantumváltozók teleportációjával. [IV]

4. Az alacsonydimenziós koherensállapot-szuperpozíciók használata hasznosnak bizonyult a fény nemklasszikus állapotainak leírásában. Ezért ezek többmódusú általánosítása, és kvantumkommunikációs elrendezésekre való alkalmazása perspektivikus terület.

   Megmutattam, hogy az egymódusú elektromágneses tér Braunstein-Kimble féle teleportációja koherensállapot-reprezentációban direkt módon tárgyalható. A felhasznált összefonódott állapotokat konjugált koherensállapot párok szuperpozíciójaként írtam fel. Ezek a kvantumklónozás elméletéből is ismertek. A szuperpozíciós integrálban elegendő négy helyett két változóban integrálni. Az alkalmazott eljárás alternatívája a többmódusú fény fázistérbeli vagy fotonszám reprezentációt használó leírásának. [III,V,VII]

5. Optikai multiportok és fotodetektálás alkalmazásával. és fotodetektálással a fény számos kvantumállapota előállítható.

   Általánosítottam egy ilyen interferometrikus elrendezést, az irodalomban ismert „kvantumollót”. Az „általánosított kvantumolló” alkalmas kevésfotonos Fock-állapot szuperpozíciók előállítására, manipulálására, egy módus első néhány fotonszám-komponensét teleportálva. Az elrendezés tervezése optimalizált paraméterű nyalábosztók felhasználását igényli. [I]

6. Ismert, hogy a $\mathfrak{su}(2)$ és a $\mathfrak{su}(3)$ Lie-algebra is megvalósítható bozonokkal. Ennek alapján a veszteségmentes nyalábosztók $SU(2)$ szimmetria segítségével írhatók le.

   Elemeztem az „általánosított kvantumolló” teleportációs aspektusát a nyalábosztók $SU(2)$ szimmetriájának felhasználásával. A szimmetria hasznosnak bizonyult az eszköz működésésnek megértéséhez. A nyalábosztók $SU(2)$-elméletével analóg leírást adtam a passziv, lineáris, veszteségmentes optikai hatportokra (tritterekre), az $SU(3)$ szimmetria felhasználásával. [I,VIII]

A bemutatott eredményekhez kapcsolódó publikációkra már érkeztek hivatkozások. Ismereteim szerint a végesdimenziós Wigner-függvényeket alkalmazó cikkemet követően egy új Wigner-függvény formalizmust vezettek be, amely jobban illeszkedik a kvantuminformációelméleti problémákhoz. A koherensállapot-szuperpozíciókat is továbbfejlesztették: teljes bázist találtak a kétmódusú fény Hilbert-terén, és a zajos környezetben történő teleportációt is leírták. A vákuum és egyfoton-állapot szuperpozíciójának teleportálása már kísérletileg is elérhető, és a kvantumolló megvalósításának lehetőségét is több szerző elemezte.

Summary

In this thesis, I have presented my results concerning quantum teleportation and photon interference. My work contributes to the field of quantum communication theory and quantum optics.

Quantum teleportation is the basic constituent of quantum communication, and one of the most interesting applications of Einstein-Podolsky-Rosen correlations. I have investigated a classical limit of quantum teleportation to benefit the understanding of the quantum case. I gave alternative descriptions of the phenomenon, utilizing several different formalisms. Each description endows new points of view, disclosing certain aspects of teleportation processes.

I have studied optical schemes, where modes of the electromagnetic field with a very low photon number interfere. Specifically, I have generalized a quantum state preparation method, which is mainly based on an optical implementation of quantum teleportation. I have found, that the well-known application of $SU(2)$ symmetry in description of beam splitters helps in understanding the operation of this interferometric scheme. Motivated by that, I described passive lossless linear optical six-ports with the aid of $SU(3)$ symmetry.

In the following I summarize the main points I have considered. The list of related publications follows this summary page.

1. I have shown, that the Bennett scheme for quantum teleportation of a qubit is the quantum mechanical generalization of the "one-time-pad" or Vernam's cipher. The latter is the theoretically most reliable way of secure classical communication, but as it requires a huge number of correlated true random bits, it is rarely used in practice.

   The understanding of the relation between the "one-time-pad" and teleportation helps understanding the nature of the quantum process. The classical-to-quantum transition is illustrated by an analysis of a gedanken experiment. [II]

2. I have considered probabilistic quantum teleportation processes utilizing pure states as entangled resource and projective measurement, on finite dimensional Hilbert-spaces. There exists a description of states of bipartite quantum systems in terms of

antilinear operators. Maximally entangled states are described by antiunitary operators in this formalism. Antiunitary operators are generally related to time reversal symmetry in physics. This is another interesting application of these operators, having several advantages, such as independence of the Hilbert-space basis. This representation is deeply related to the relative state representation of quantum states and channels. Utilizing the antilinear representation, I have expressed quantum teleportation as a quantum operation or quantum channel in a compact, convenient form. [VI]

3. A prevalent alternative description of quantum phenomena is the application of quasiprobability distributions. One of the most frequently used quasiprobability distribution is the Wigner function. Quantum teleportation of continuous variables (i.e. oscillator states) was first described in this formalism, while Bennett's original quantum teleportation scheme was introduced in terms of quantum states directly, on a finite dimensional Hilbert-space.

   There are several ways of defining Wigner functions for finite dimensional Hilbert spaces. Choosing one of these formalisms, I have described Bennett's quantum teleportation in terms of discrete Wigner-functions. This new description clearly demonstrates the process of teleportation in the discrete phase space, revealing the connection with teleportation of continuous quantum variables. This was one of the first applications of discrete Wigner functions in quantum information context. [IV]

4. Low-dimensional coherent-state representations have proven to be useful in describing nonclassical states of light. Their multimode generalization and application to quantum communication arrangements is therefore tempting.

   I have shown, that the Braunstein-Kimble teleportation scheme for teleportation of a single-mode electromagnetic field can be described in terms of coherent-state superpositions in a direct way. The entangled states utilized in the process are described by means of conjugate coherent state pairs, which also appear in the theory of quantum cloning. In the superposition integral it suffices to integrate over two real dimensions instead of four. The method applied for this is an alternative to the number-state or phase-space representation method in the treatment of multi-mode fields. [III,V,VII.]

5. By application of optical multiports (arrangements of beam-splitters and phase shifters) and photodetection, it is possible to prepare certain quantum states of light.

   I have presented such an interferometric scheme, the "generalized quantum scissors", which is capable of generating superpositions of few-photon Fock-states. The device teleports the first few Fock-components of an electromagnetic field mode. Its design relies on the application of beam splitters with optimized parameters. [I]

6. It is known, that both $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ Lie-algebras have bosonic realizations. Based on this, on may describe beam-splitters in terms of $SU(2)$ symmetry.

   I have analyzed the teleportation aspect of the "generalized quantum scissors" by exploiting $SU(2)$ symmetry of beam splitters. That turned out to be useful in understanding the operation of the device. I also gave a description of passive, lossless, linear optical six-ports (tritters) in terms of $SU(3)$ symmetry, in analogy with $SU(2)$ theory of beam splitters. [I,VIII]

These results have already received some citations. Up to my knowledge, my pioneering Wigner-function description of teleportation led to an introduction of an alternative finite dimensional Wigner function concept, which is more convenient in quantum information purposes. The application of coherent-state superpositions was further developed: a complete basis was found to describe arbitrary two-mode states, and a description of quantum teleportation in a noisy environment was given. The teleportation of a superposition of vacuum and one-photon state has been realized recently, and the possibility of experimental realization of quantum scissors was also investigated by several authors since.

List of related publications

I. M. Koniorczyk, Z. Kurucz, A. Gábris and J. Janszky: *General optical state truncation and its teleportation*, Phys. Rev. A **62**, 013802 (2000).

II. M. Koniorczyk, T. Kiss, and J. Janszky: *Teleportation: from probability distributions to quantum states*, J. Phys. A (Math. Gen.) **34** pp. 6949-6955 (2001). `arXive:quant-ph/0011083`

III. J. Janszky, M. Koniorczyk and A. Gábris: *One-complex-plane representation approach to quantum teleportation*, Phys. Rev. A **64** 034302 (2001).

IV. M. Koniorczyk, V. Bužek, and J. Janszky: *Wigner-function description of quantum teleportation in arbitrary dimensions and continuous limit*, Phys. Rev. A **64** 034301 (2001). `arXive:quant-ph/0106109`

V. J. Janszky, A. Gábris, M. Koniorczyk, A. Vukics and P. Adam: *Coherent-state approach to entanglement and teleportation*, Progress of Physics, **49** pp. 993-1000 (2001).

VI. Z. Kurucz, M. Koniorczyk, and J. Janszky: *Teleportation with partially entangled states*, Progress of Physics **49** pp. 1019-1025 (2001).

VII J. Janszky, A. Gábris, M. Koniorczyk, A. Vukics and J. Asboth: *One-complex-plane representation: a coherent-state description of entanglement and teleportation*, J. Opt. B.: Quantum Semiclasss. Opt. **4**, pp. S213-S217 (2002).

VIII. M. Koniorczyk and J. Janszky: *Photon number conservation and photon interference*, invited paper in "First International Workshop on Classical and Quantum Interference", Jan Peřina, Miroslav Hrabovský and Jaromir Křepelka Editors, Proceedings of SPIE **4888**, pp 1-8. (2002). `arXive:quant-ph/0110170`

Acknowledgements

I would like to thank all the people I have worked together during the past few years. First of all my supervisor, Professor József Janszky, who introduced me to the world of research, and gave me a lot of knowledge and inspiration. I would also like to thank the people of Quantum Optics group at Research Institute of Solid State Physics and Optics, namely Péter Ádám, Péter Domokos, Aurél Gábris, Attila Kárpáti, Zsolt Kis, Tamás Kiss, Zoltán Kurucz, and Szilárd Szabó. Special thanks to János Asbóth, András Vukics, and Géza Tóth for reading the manuscript utmost thoroughly. I thank to Professor Vladimír Bužek and his group at Quantum Information group of the Slovak Academy of Sciences, Bratislava. I thank the collaboration and discussion to Ákos László, the young mathematician of the Department of Mathematics, University of Pécs, who is not amongst us anymore. Finally, I thank to my friends in Pécs, namely László Pere, Győző Harka and Titusz Bugya for inspiring discussions.

Some of the notations

$\mathbb{N}$	set of natural numbers $(0,1,2,\ldots)$
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$\mathscr{H}$	Hilbert space
span	vector space spanned by the vectors in argument
$\hat{1}$	identity operator
$SO(n)$	group of n-dimensional rotations
$SU(n)$	Lie group of $n \times n$ unimodular unitary matrices of unit determinant
$\mathfrak{su}(n)$	Lie algebra corresponding to $SU(n)$
$\delta_{i,j}$	Kronecker-delta
$\delta(x)$	Dirac-delta
$\times$	Cartesian product of sets: $U \times V = \{(u,v) \| u \in U,\ v \in V\}$
$\circ$	composition of functions
*	complex conjugate
†	Hermitian conjugate
*	adjoint of antiunitary operator
$\|\|\ldots\rangle$	quadrature eigenstate

Bibliography

[1] P. Adam, I. Földesi, and J. Janszky. Complete basis set via straight-line coherent-state superpositions. *Phys. Rev. A*, 49(2):1281–1287, February 1994.

[2] R. Arens and V. S. Varadarajan. On the concept of Einstein-Podolsky-Rosen states and their structure. *J. Math. Phys.*, 41(2):638–651, February 2000.

[3] S. M. Barnett and D. T. Pegg. Phase measurement by projection synthesis. *Phys. Rev. Lett.*, 76(22):4148–4150, May 1996.

[4] S. M. Barnett and D. T. Pegg. Optical state truncation. *Phys. Rev. A*, 60(6):4965–4973, December 1999.

[5] J. S. Bell. On the Einstein-Podolsky-Rosen paradox. *Physics*, 1:195–200, 1964.

[6] J. S. Bell. EPR correlations and EPW distributions. *Ann. (N.Y.) Acad. Sci.*, pages 263–266, 1986.

[7] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters. Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels. *Phys. Rev. Lett.*, 70(13):1895–1999, March 1993.

[8] P. Bianucci, C. Miquel, J. P. Paz, and M. Saraceno. Discrete Wigner functions and the phase space representation of quantum computers. *Phys. Lett. A*, 297(5-6):353–358, 2002.

[9] L. C. Biedenharn and J. D. Louck. *Angular momentum in quantum physics*. Addison-Wesley, 1981.

[10] D. Bohm. A suggested interpretation of the quantum theory in terms of “hidden” variables. i. *Phys. Rev.*, 85(2):166–179, January 1952.

[11] D. Bohm. A suggested interpretation of the quantum theory in terms of “hidden” variables. ii. *Phys. Rev.*, 85(2):180–193, January 1952.

[12] D. Boschi, S. Branca, F. D. Martini, L. Hardy, and S. Popescu. Experimental realization of teleporting an unknown pure quantum state via dual classical and Einstein-Podolsky-Rosen channels. *Phys. Rev. Lett.*, 80(6):1121–1125, February 1998.

[13] S. Bose and V. Vedral. Mixedness and teleportation. *Phys. Rev. A*, 61:040101(R), March 2000.

[14] A. N. Boto, P. Kok, D. S. Abrams, S. L. Braunstein, C. P. Williams, and J. P. Dowling. Quantum interferometric optical lithography: Exploiting entanglement to beat the diffraction limit. *Phys. Rev. Lett.*, 85(13):2733–2736, September 2000.

[15] D. Bouwmeester, J.-W. Pan, K. Mattle, M. Eibl, H. Weinfurter, and A. Zeilinger. Experimental quantum teleportation. *Nature*, 390:575–579, December 1997.

[16] S. L. Braunstein, V. Bužek, and M. Hillery. Quantum information distributors: Quantum network for symmetric and asymmetric cloning in arbitrary dimension and continuous limit. *e-print*, 2000. quant-ph/0009076.

[17] S. L. Braunstein and H. J. Kimble. Teleportation of continuous quantum variables. *Phys. Rev. Lett.*, 80(4):869–872, January 1998.

[18] S. L. Braunstein and H. J. Kimble. Dense coding for continuous variables. *Phys. Rev. A*, 61:042302, March 2000.

[19] V. Bužek, A. D. Wilson-Gordon, P. L. Knight, and W. K. Lai. Coherent states in a finite-dimensional basis: their phase properties and relationship to coherent states of light. *Phys. Rev. A*, 45(11):8079–8094, June 1992.

[20] M. Byrd. The geometry of $SU(3)$. *e-print*, 1997. physics/9708015.

[21] M. Byrd and E. C. G. Sudarshan. $SU(3)$ revisited. *J. Phys. A-Math. Gen.*, 31:9255–9268, 1998.

[22] M. S. Byrd and E. C. G. Sudarshan. $SU(3)$ revisited. *e-print*, 1998. physics/9803029.

[23] K. E. Cahill. Coherent-state representations for the photon density operator. *Phys. Rev.*, 138(6B):B1566–B1576, 1965.

[24] R. A. Campos, B. E. A. Saleh, and M. C. Teich. Quantum-mechanical lossless beam splitter: $SU(2)$ symmetry and photon statistics. *Phys. Rev. A*, 40(3):1371–1384, August 1989.

[25] N. J. Cerf and S. Iblisdir. Phase-conjugated inputs quantum cloning machines. *e-print*, 2001. quant-ph/0102077.

[26] N. J. Cerf and S. Iblisdir. Phase conjugation of continuous quantum variables. *e-print*, 2001. quant-ph/0012020.

[27] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt. Proposed experiment to test local hidden-variable theories. *Phys. Rev. Lett.*, 23(15):880–884, October 1969.

[28] C. Cohen-Tanoudji, B. Diu, and F. Laoë. *Quantum Mechanics*. Wiley-Interscience, 1998.

[29] M. Dakna, J. Clausen, L. Knöll, and D.-G. Welsch. Generation of arbitrary quantum states of travelling fields. *Phys. Rev. A*, 59(2):1658–1661, February 1999.

[30] G. M. D'Ariano, P. L. Presti, and M. F. Sacchi. Bell measurements and observables. *Phys. Lett. A*, 272:32–38, July 2000.

[31] A. Einstein, B. Podolsky, and N. Rosen. Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.*, 47(10):777–780, May 1935.

[32] J. Fiurašek. Optical implementation of continuous-variable quantum cloning machines. *Phys. Rev. Lett.*, 86(21):4942–4945, May 2001.

[33] V. Fock. Verallgemeinerung und Lösung der Diracschen statistischen Gleichung. *Z. Phys.*, 49:339–357, 1928.

[34] A. Furusawa, J. L. Sorensen, S. L. Braunstein, C. A. Fuchs, H. J. Kimble, and E. S. Polzik. Unconditional quantum teleportation. *Science*, 282:706–709, October 1998.

[35] N. Gisin. Nonlocality criteria for quantum teleportation. *Phys. Lett. A*, 210:157–159, January 1996.

[36] R. J. Glauber. The quantum theory of optical coherence. *Phys. Rev.*, 130(6):2529–2539, June 1963.

[37] M. Hillery, V. Bužek, and M. Zimán. Probabilistic implementation of universal quantum processors. *e-print*, 2001. quant-ph/0106088.

[38] H. F. Hofmann, T. Ide, T. Kobayashi, and A. Furusawa. Fidelity and information in the teleportation of continuous quantum variables. *Phys. Rev. A*, 62:062304, November 2000.

[39] M. Horodecki, P. Horodecki, and R. Horodecki. General teleportation channel, singlet fraction and quasidistillation. *Phys. Rev. A*, 60(3):1888–1898, September 1999.

[40] J. D. Jackson. *Classical Electrodynamics*. John Wiley & Sons, 3rd edition, 1998.

[41] J. Janszky, P. Domokos, and P. Adam. Coherent states on a circle and quantum interference. *Phys. Rev. A*, 48(3):2213–2219, September 1993.

[42] J. Janszky, P. Domokos, S. Szabó, and P. Adam. Quantum-state engineering via discrete coherent-state superpositions. *Phys. Rev. A*, 51(5):4191–4193, May 1995.

[43] J. Janszky, A. Gábris, M. Koniorczyk, A. Vukics, and P. Adam. Coherent-state approach to entanglement and teleportation. *Fortschr. Phys.*, 49(10-11):993–1000, October 2001.

[44] J. Janszky, M. Koniorczyk, and A. Gábris. One-complex-plane representation approach to continuous variable quantum teleportation. *Phys. Rev. A*, 64:034302, September 2001.

[45] J. Janszky and A. V. Vinogradov. Squeezing via one-dimensional distribution of coherent states. *Phys. Rev. Lett.*, 64(23):2771–2774, June 1990.

[46] D. Kaszlikowski, P. Gnaciński, M. Żukowski, W. Miklaszewski, and A. Zeilinger. Violations of local realism by two entangled n-dimensional systems are stronger than for two qubits. *Phys. Rev. Lett.*, 85(21):4418–4421, November 2000.

[47] Y.-H. Kim, S. P. Kulik, and Y. Shih. Quantum teleportation of a polarization state with a complete Bell state measurement. *Phys. Rev. Lett.*, 86(7):1370–1373, February 2001.

[48] E. Knill, R. Laflamme, and G. J. Milburn. A scheme for efficient quantum computation with linear optics. *Nature*, 409:46–52, 2001.

[49] P. Kok and S. L. Braunstein. Detection devices in entanglement-based optical state preparation. *Phys. Rev. A*, 63:033812, March 2001.

[50] Z. Kurucz, M. Koniorczyk, and J. Janszky. Teleportation with partially entangled states. *Fortschr. Phys.*, 49(10-11):1019–1025, October 2001.

[51] U. Leonhardt. Quantum-state tomography and discrete Wigner function. *Phys. Rev. Lett.*, 74(21):4101–4105, May 1995.

[52] U. Leonhardt. *Measuring the Quantum State of Light*. Cambridge University Press, 1997.

[53] H. J. Lipkin. *Lie groups for pedestrians*. North-Holland Publishing CO., Amsterdam, 1965.

[54] A. Lukš and V. Peřinová. Ordering of ladder operators, the Wigner function for number and phase, and the enlarged hilbert-space. *Phys. Scr.*, T48:94–99, 1993.

[55] N. Lütkenhaus, J. Calsamiglia, and K. A. Suominen. Bell measurements for teleportation. *Phys. Rev. A*, 59(5):3295–3300, May 1999.

[56] G. J. Milburn and S. L. Braunstein. Quantum teleportation with squeezed vacuum states. *Phys. Rev. A*, 60(2):937–942, August 1999.

[57] M. Nielsen and C. M. Caves. Reversible quantum operations and their application to teleportation. *Phys. Rev. A*, 55(4):2547–2556, april 1997.

[58] T. Opatrný, V. Bužek, J. Bajer, and G. Drobný. Propensities in discrete phase spaces: Q function of a state in a finite-dimensional Hilbert space. *Phys. Rev. A*, 52(3):2419–2428, September 1995.

[59] T. Opatrný, G. Kurizki, and D.-G. Welsch. Improvement on teleportation of continuous variables by photon substraction via conditional measurement. *Phys. Rev. A*, 61:032302, February 2000.

[60] H. Paul, P. Törmä, T. Kiss, and I. Jex. Photon chopping: new way to measure the quantum state of light. *Phys. Rev. Lett.*, 76(14):2464–2467, April 1996.

[61] J. P. Paz. Discrete Wigner functions and the phase-space representation of quantum teleportation. *Phys. Rev. A*, 65(6):062311, 2002.

[62] D. T. Pegg, L. S. Phillips, and S. M. Barnett. Optical state truncation by projection synthesis. *Phys. Rev. Lett.*, 81(8):1604–1606, August 1998.

[63] A. Peres. *Quantum Theory : Concepts and Methods (Fundamental Theories of Physics, Vol 57)*. Kluwer Academic Publishers, 1995.

[64] J. Preskill. Lecture notes for physics 229: Quantum information and computation. available at `http://www.theory.caltech.edu/people/preskill`, September 1998.

[65] P. M. Radmore and S. M. Barnett. *Methods in Theoretical Quantum Optics*. Oxford University Press, 1997.

[66] M. Reck and A. Zeilinger. Experimental realization of any discrete unitary operator. *Phys. Rev. Lett.*, 73(1):58–61, July 1994.

[67] B. Schneier. *Applied Cryptography*. John Wiley & Sons, New York, 2nd edition, 1996.

[68] B. Schumacher. Sending entanglement through noisy quantum channes. *Phys. Rev. A*, 54(4):2614–2628, October 1996.

[69] O. Steuernagel and J. A. Vaccaro. Reconstructing the density operator via simple projectors. *Phys. Rev. Lett.*, 75(18):3201–3205, October 1995.

[70] U. M. Titulaer and R. J. Glauber. Density operators for coherent fields. *Phys. Rev.*, 145(4):1041–1049, May 1966.

[71] J. Vaccaro. Number-phase Wigner function on Fock space. *Phys. Rev. A*, 52(5):3474–3488, November 1995.

[72] J. A. Vaccaro and D. T. Pegg. Wigner-function for number and phase. *Phys. Rev. A*, 41(9):5156–5163, May 1990.

[73] L. Vaidman. Teleportation of quantum states. *Phys. Rev. A*, 49(2):1473–1476, February 1994.

[74] S. J. van Enk. Discrete formulation of teleportation of continuous variables. *Phys. Rev. A*, 60(6):5095–5097, December 1999.

[75] P. Varga. On the possibility of detection of a two-photon state by a single detector. In *Proceedings of the Second Central European Workshop on Quantum Optics*, pages 33–35, Budapest, 1994. Research Laboratory for Crystal Physics.

[76] C. J. Villas-Bôas, N. G. de Almeida, and M. H. Y. Moussa. Teleportation of zero and one-photon running-wave states by projection synthesis. *Phys. Rev. A*, 60(4):2759–2763, October 1999.

[77] A. Vukics, J. Janszky, and T. Kobayashi. Nonideal teleportation in a coherent-state basis. *Phys. Rev. A*, 66:023809, August 2002.

[78] R. F. Werner. Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model. *Phys. Rev. A*, 40(8):4277–4280, October 1989.

[79] K. Wodkiewcz, P. L. Knight, S. J. Buckle, and S. M. Barnett. Squeezing and superposition states. *Phys. Rev. A*, 35(6):2567–2577, March 1987.

[80] W. K. Wootters. A Wigner-function formulation of finite state quantum mechanics. *Annals of Physics*, 176:1–21, 1987.

[81] W. K. Wootters and W. H. Żurek. A single quantum cannot be cloned. *Nature*, 299:802–803, 1982.

[82] A. Yariv. *Quantum Electronics*. John Wiley & Sons, 3rd edition, 1989.

[83] S. Yu and C.-P. Sun. Canonical quantum teleportation. *Phys. Rev. A*, 61:022310, January 2000.